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THE MATHEMATICS TEACHER

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ISSUES
1&2

THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

The Association of Mathematics Teachers of India (AMTI) was started in 1965 for the promotion of efforts to improve Mathematics education at all levels. A major aim of the Association is to assist practising teachers of Mathematics in schools in improving their expertise and professional skills. Another important aim is to spot out and foster Mathematical talents in the young. The Association also seeks to disseminate new trends in Mathematics education among parents and public. Other activities of the Association include consultancy services to schools in equipping the Mathematics section of their libraries, in organising children's Mathematics clubs and fairs, in setting up teacher centres in schools, in conducting Mathematics laboratory programmes, in holding practical tests in Mathematics in assisting children in participating investigational projects.

The Association holds " National Mathematical Talent Search Competition " annually and organizes Orientation Courses, Seminars and Workshops for teachers and courses for talented students. A national conference is held annually in different parts of the country for teachers to meet and deliberate on important issues of Mathematics education. Innovative teacher award has been instituted to give public recognition to enterprising and pioneering teachers of Mathematics for which entries from teachers are invited.

. An award for contributions to the Mathematics Teacher relating to History of Mathematics in the context of mathematics education had been instituted by Prof.R.C.Gupta.

"The Mathematics Teacher (India)" (MT) is the official quarterly journal of the Association and is issued twice a year. It has been approved for use in schools and colleges of education by the Government departments of education in many States. Besides MT the Association also brings out Junior Mathematician (JM), three issues in a year, especially for school students in English and Tamil.

The membership of the Association is open to all those interested in Mathematics and Mathematics Education. The membership fee inclusive of subscription for "**The Mathematics Teacher (India)**" and effective from April 1993 is as follows:

Subscription for India*

Category	Individual	Institutional
Life	Rs. 500	Rs. 1000
Annual (Ordinary)	Rs. 50	—
Junior Mathematician - Life	Rs. 250	Rs. 250
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The Journal "The Mathematics Teacher" will be supplied free to all members.
Fifty or more subscriptions to Junior Mathematician will entail 20% discount.

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(INDIA)

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Prof. Hemalatha Thyagarajan
Editor

EDITORIAL

With great difficulty this issue is being released thanks to the voluntary but resourceful help offered by Dr. R. Bhaskaran of Madurai. The activities of the association are increasing manifold with the same few volunteers in charge. We hope to receive your feedback to enthuse us further.

The Association of Mathematics Teachers of India
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E-mail [illegible] *Website* [illegible]

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SCREENING TEST- GAUSS CONTEST

(NMTC at PRIMARY LEVEL V to VI Standards)

PART - A

1. a, b where $a > b$ are natural numbers each less than 10 such that $(a^2 - b^2)$ is a prime number. The number of such pairs (a, b) is

   (A) 5 (B) 6 (C) 7 (D) 8

   Solution: Since a, b are each less than 10. It is easy to enumerate the pairs are (2,1), (3,2), (4,3), (5,4), (6,5), (7,6), (8.7), (9.8). These pairs are chosen because $a^2 - b^2$ is prime $\Rightarrow (a-b)(a+b)$ is a prime $\Rightarrow a - b = 1$.

   Out of these pairs $5^2 - 4^2 = 25 - 16 = 9$ and $8^2 - 7^2 = 15$ are composite. The other pairs gives $2^2 - 1^2 = 3, 3^2 - 2^2 = 5$, $4^2 - 3^2 = 7, 6^2 - 5^2 = 11$, $7^2 - 6^2 = 13, 9^2 - 8^2 = 17$ which are all primes. $\therefore$ There are 6 pairs.

   Answer: (B).

2. The number of three digit numbers that are divisible by 2 but not divisible by 4 is

   (A) 200 (B) 225 (C) 250 (D)450

   Solution: Total number of 3 digit numbers $= 999 - 99 = 900$. Every second number starting from 100 is divisible by 2 (even numbers).

   The numbers of even numbers $= \frac{900}{2} = 450$. Out of these 450 numbers every second number is divisible by 4,

   Number of such numbers $= \frac{450}{2} = 225$.

   $\therefore$ Number of numbers divisible by 2 but not by 4 is equal to $450 - 225 = 225$.

   Answer: (B).

3. A, B, C are single digits. In this multiplication B could be

$$\begin{array}{r} AB\times \\ \quad 7 \\ \hline BCA \end{array}$$

(A) 7 (B) 1
(C) 2 (D) 4

Solution: If $B = 7$ then $\begin{array}{r} A7 \\ \times\ 7 \\ \hline \therefore\ 9 \end{array} \Rightarrow A = 9$.

The number is 97 and $97 \times 7 = 679$.

The hundreds digit is $6 \neq B$.

If $B = 1$ then $\begin{array}{r} A1 \\ \times\ 7 \\ \hline \therefore\ 7 \end{array} \Rightarrow A = 7$.

The number is 71 and $71 \times 7 = 497$.

The hundreds digit is $4 \neq B$.

If $B = 4$ then $\begin{array}{r} A4 \\ \times\ 7 \\ \hline \therefore\ 8 \end{array} \Rightarrow A = 8$.

The number is 84. $\therefore$ $84 \times 7 = 588$.

The hundreds digit is $5 \neq B$.

If $B = 2$ then $\begin{array}{r} A2 \\ \times\ 7 \\ \hline \therefore\ 4 \end{array} \Rightarrow A = 4$.

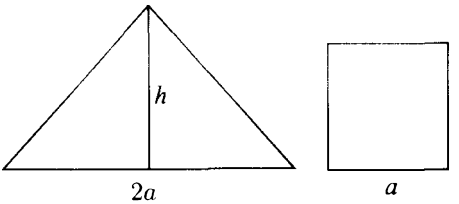
The number is 42, then $42 \times 7 = 294$.

In this case hundreds digit is $2 = B$.

Answer: (C).

4. The base of a triangle is twice as long as a side of a square. Their areas are equal. Then the ratio of the altitude of the triangle to this base to the side of the square is

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) 1 (D) 2



Solution: Let 'a' be the side of square. The base of the triangle $= 2a$. Let h be the altitude to the base $2a$.

$$\text{Given } a^2 = \frac{1}{2} \cdot 2ah$$

$$a^2 = ah \Rightarrow h = 1 \Rightarrow \frac{h}{a} = 1.$$

Answer: (C).

5. Two sequences S_1 and S_2 are as under:

$S_1 : \frac{2}{1}, \frac{4}{3}, \frac{6}{5}, \cdots$

$S_2 : \frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \cdots$

The n^{th} term of S_1 is $S_1 : \frac{2n}{2n-1}$ and the n^{th} tern of S_2 is $\frac{2n-1}{2n}$. The value of the difference between the 2012^{th} terms of S_1 and S_2 is

(A) $\frac{4023}{2012\times2011}$ (B) $\frac{8047}{4024\times4023}$ (C) $\frac{4023}{4024\times4023}$ (D) $\frac{8047}{2012\times2011}$

Solution: Required value $= \frac{2n}{2n-1} - \frac{2n-1}{2n} = \frac{4n^2-(2n-1)^2}{2n(2n-1)}$

$= \frac{4n-1}{2n(2n-1)} = \frac{4(2012)-1}{2(2012)[2(2012-1)]}$ (by putting $n = 2012$)

$= \frac{8048-1}{2(2012)(4024-1)} = \frac{8047}{4024\times4023}$.

Answer: (B).

6. The least number which when divided by 25, 40 and 60 leaves a remainder 7 in each case is

(A) 607 (B) 1007 (C) 807 (D) 507

Solution: The LCM of the three numbers 25,40 and 60 is found

2	25,40,60
2	25,20,30
5	25,10,15
	5, 2, 3

LCM of 25,40, 60 is $2 \times 2 \times 5 \times 5 \times 2 \times 3 = 600$. Since the number leaves a remainder 7 in each case the required number is $= 600 + 7 = 607$.

Answer: (A).

7. The integers greater than 1 are arranged in 5 columns as follows.

	Column (1)	Column (2)	Column (3)	Column (4)	Column (5)
Row 1 →		2	3	4	5
Row 2 ←	9	8	7	6	
Row 3 →		10	11	12	13
Row 4 ←	17	16	15	14	
⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮

In the odd numbered rows, the integers appear in the last 4 columns are increasing form left to right. In the even numbered rows, the integers appear in the first four columns are increasing from right to left. In which column will the number 2012 appear?

(A) fourth (B) second (C) first (D) fifth

Solution:

	C_1 (1)	C_2 (2)	C_3 (3)	C_4 (4)	C_5 (5)
R_1		2	3	4	5
R_2	9	8	7	6	
R_3		10	11	12	13
R_4	17	16	15	14	
R_5		18	19	20	21
R_6	25	24	23	22	
R_7		26	27	28	29
R_8	33	32	31	30	

The numbers in C_1, C_3 and C_5 are all odd. $\therefore$ 2012 cannot be in any of these column so 2012 may be in either C_2 or C_4.

The elements in C_2 (except 2) are all divisible by 8 alternatively starting with 8 and not divisible by 4 alternatively starting with 10. But 2012 is not divisible by 8 and divisible by 4. $\therefore$ 2012 can not be in C_2.

Answer: The fourth column (A).

8. *Akash, Bharath, Christe, Daniel* and *Eashwar* are friends. The interesting fact is that all of them were born in the same year, but on different days, different dates and different months. If *Akash* were born on February 19, then Daniel could have been born on

(A) March 30 (B) August 20 (C) December 25 (D) April 16

Solution:

Case (i) Let us consider the year to be non leap year.

Let February 19 be a Tuesday.

The other Tuesdays in that year for various months can be got by adding 7 days to Feb 19. They are:

Feb	26
March	5, 12, 19 , 26
April	2, 9, $\boxed{16}$, 23, 30
May	7, 14, 21, 28
June	4, 11, 18, 25
July	2, 9, 16, 23, 30
August	6, 13, $\boxed{20}$, 27
September	3, 10, 17, 24
October	1, 8, 15, 22, 29
November	5, 12, 19, 26
December	3, 10, 17, 24

The option (B) and (D) namely August 20 and April 16 are Tuesdays. But March 30^{th} and December 25^{th} are not Tuesdays. $\therefore$ Correct option is (A) or (C).

Answer: (A) or (C).

Case (ii) Let us consider the year to be a leap year.

Let February 19 be a Sunday.

The other Sundays in that year for various months can be got by adding 7 days Feb 19. They are:

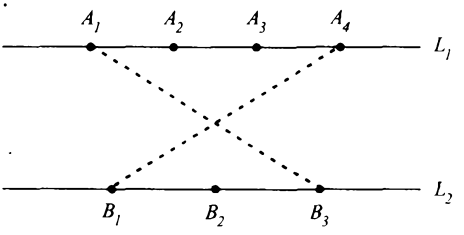
Feb	26
March	4, 11, 18 , 25
April	1, 8, 15, 22, 29
May	6, 13, 20, 27
June	3, 10, 17, 24
July	1, 8, 15, 22, 29
August	5, 12, 19, 26
September	2, 9, 16, 23, 30
October	7, 14, 21, 28
November	4, 11, 18, 25
December	2, 9, 16, 23, 30

(A) March 30 $\rightarrow$ Friday
(B) August 20 $\rightarrow$ Monday
(C) December 25 $\rightarrow$ Tuesday
(D) April 16 $\rightarrow$ Monday

$\therefore$ Correct options are

Answer: (A) or (B) or (C) .

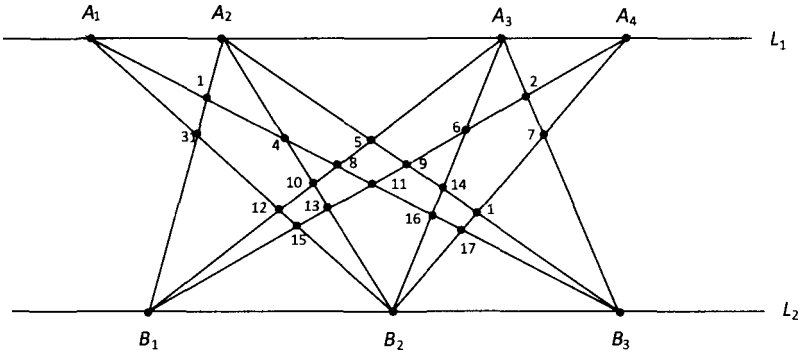
9. In the adjoining figure points A_1, A_2, A_3, A_4 are located on the line L_1 and B_1, B_2, B_3 are located on the line L_2.



Each one of the points on L_1 is connected to each one of the point on L_2. (Example A_1 to B_3 and A_4 to B_1 as in the figure). The line segments are not extended. No line segment passes through the point of intersection of any two lines segments. The number of points of inter section of all these line segments is (Exclusive of A_1, A_2, A_3, A_4 and B_1, B_2, B_3).

(A) 15 (B) 16 (C) 17 (D) 18

Solution: Clearly A_1B_1 and A_4B_3 cannot intersect with any of the other lines internally.



By drawing the figure it is seen that the maximum number of print of intersection is 18. At the primary level it is expected of the students to enumerate by drawing the figure.

Answer: (D).

10. A box contains 100 balls of different colours: 28 *Red*, 17 *Blue*, 21 *Green*, 10 *white*, 12 *Yellow* and 12 *Black*. The smallest number of balls drawn from the box so that at least 15 balls are of the same colour is

(A) 73 (B) 77 (C) 81 (D) 85

Solution: When a person starts taking the balls without seeing the colour he/she may get the following combination:

10 White
12 Yellow
12 Black
14 Red
14 Blue
14 Green

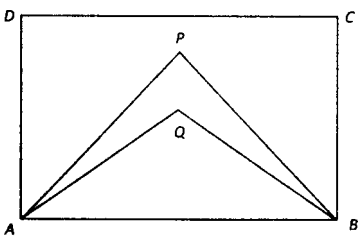
Now only Red, Blue and Green balls are remaining. If the person taken any one of these balls then 15 balls of the same colour be obtained.

$\therefore$ The minimum number of balls to be taken is 77 when we want 15 balls of the same colour to be obtained.

Answer: (B).

PART - B

1. $ABCD$ is a rectangle. AP, AQ divide $\angle DAB$ in to three equal parts and BP and BQ divide $\angle CBA$ into three equal parts.



If $k(\angle APB) = (\angle AQB)$ then the value of k is —— .

Solution:

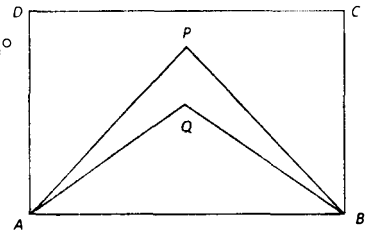
$$\begin{aligned} \angle QAB = \angle QBA &= \frac{90}{3} = 30^\circ \\ \angle QAB + \angle QBA &= 30 + 30 = 60^\circ \\ \therefore \quad \angle AQB &= 180^\circ - 60^\circ = 120^\circ \end{aligned}$$

Similarly

$\angle PAB + \angle PBA = 60 + 60 = 120^\circ$
$\angle APB = 180 - 120 = 60^\circ$

$k(\angle APB) = m\angle AQB$

$k(60) = 120^\circ \Rightarrow k = 2.$



2. Here is a sequence of composite numbers having only one prime factor, written in ascending order 4,8,9,16,25,27,32, $\cdots$. The 15^{th} number of this sequence is —— .

Solution: Since the numbers of the sequence are composite and should be any power of a prime number as the terms having only one prime factor we have

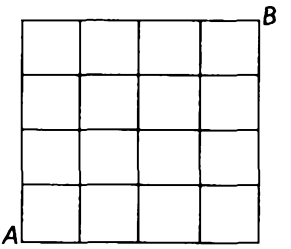
First term	t_1	$=$ 4	$=$	2^2	$\rightarrow$	$(\text{prime})^2$
	t_2	$=$ 8	$=$	2^3	$\rightarrow$	$(\text{prime})^3$
	t_3	$=$ 9	$=$	3^2	$\rightarrow$	$(\text{prime})^2$
	t_4	$=$ 16	$=$	2^4	$\rightarrow$	$(\text{prime})^4$
	t_5	$=$ 25	$=$	5^2	$\rightarrow$	$(\text{prime})^2$
	t_6	$=$ 27	$=$	3^3	$\rightarrow$	$(\text{prime})^3$
	t_7	$=$ 32	$=$	2^5	$\rightarrow$	$(\text{prime})^5$
	t_8	$=$ 49	$=$	7^2	$\rightarrow$	$(\text{prime})^2$
	t_9	$=$ 64	$=$	2^6	$\rightarrow$	$(\text{prime})^6$
	t_{10}	$=$ 81	$=$	3^4	$\rightarrow$	$(\text{prime})^4$
	t_{11}	$=$ 121	$=$	11^2	$\rightarrow$	$(\text{prime})^2$
	t_{12}	$=$ 125	$=$	5^3	$\rightarrow$	$(\text{prime})^3$
	t_{13}	$=$ 128	$=$	2^7	$\rightarrow$	$(\text{prime})^7$
	t_{14}	$=$ 169	$=$	13^2	$\rightarrow$	$(\text{prime})^2$
	t_{15}	$=$ 243	$=$	3^5	$\rightarrow$	$(\text{prime})^5$

15^{th} term is 243.

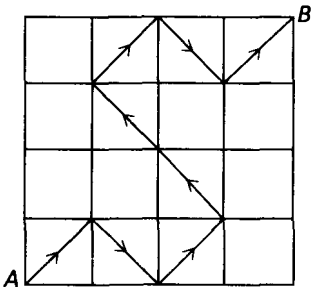
Answer: 243.

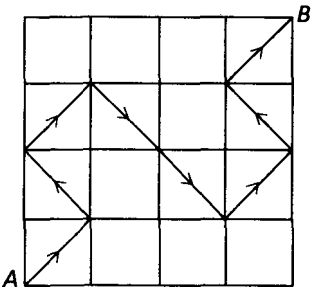
3. An insect crawls from A to B along a square lamina which is divided by lines as shown into 16 equal squares. The insect

always travels diagonally from one corner of a square to the other corner. While going it never visits the same corner of any square. If one diagonal of a smallest square is taken as 1 unit, the maximum length of the path travelled by the insect is —— .



Solution:





Maximum length of the path travelled by this insect is 8.

4. A says: "I am a 6-digit number and all my middle digits are made of zeros." B says to A: "I am your successor. My digit in the tens place is the same as your starting digit". The value of the whole number A is —— .

Solution:

A is $a\,0\,0\,0\,0\,b$.

B is 1 more than $a\,0\,0\,0\,0\,b$.

If b were 1 through 8 than 1 more gives a single digit number and the tens digit of B remains 0 which can not be a starting digit of A $\because$ $a \neq 0$.

$\therefore$ $b = 9$ $\quad\therefore$ A is $a\,0\,0\,0\,0\,9$.

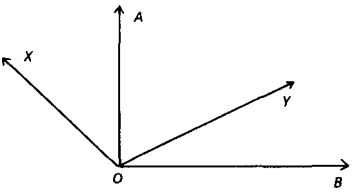
B is $a\,0\,0\,0\,0\,9 + 1 = a\,0\,0\,0\,1\,0$.

By data $\therefore\ a = 1$. $\therefore\ A$ is $1\,0\,0\,0\,0\,9$.

5. In the figure $\angle XOY = \angle AOB = 90°$.

The measure of $\angle XOB = 126°$.

The measure of $\angle AOY$ is —— .



Solution: $\angle XOB =$

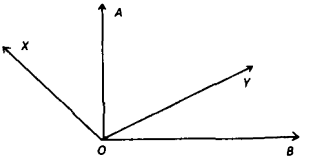
$\angle XOY + \angle YOB = 126°$

$90 + \angle YOB = 126° \Rightarrow \angle YOB = 36°$

$\angle AOB = 90°$

$\angle AOY + \angle YOB = 90°$

$\angle AOY + 36 = 90° \Rightarrow \angle YOB = 54°$



Answer: 54°

6. 6 men can do a work in 1 year and 2 months. Then 3 men can do the work in —— months.

Solution:

M	time	
6	14	time $x = \frac{6}{3} \times 14 = 28$ months
3	x	

Answer: 28 months

7. The first term of a sequence of fractions is $\frac{3}{1}$ and the n^{th} term t_n of the sequence is equal to

$$\frac{\text{sum of the numerator and denominator of } t_{n-1}}{\text{Difference of the numerator and denominator of } t_{n-1}}.$$

(Ex: If $t_1 = \frac{a}{b}$ then $t_2 = \frac{a+b}{a-b}$). The sum of this sequence to 2012 terms is —— .

Solution: If $t_1 = \frac{a}{b}$ then $t_2 = \frac{a+b}{a-b}$ $\cdots$

$t_1 = \frac{3}{1} \Rightarrow t_2 = \frac{3+1}{3-1} = \frac{4}{2} = \frac{2}{1}$

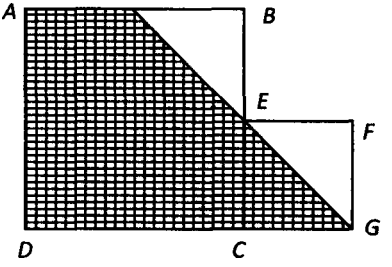
$t_3 = \frac{2+1}{2-1} = \frac{3}{1} = 3 \quad t_4 = \frac{3+1}{3-1} = 2$ $\cdots$

So the sequence is $3, 2, 3, 2, 3, \cdots$. The number of terms to be taken is $2012 \Rightarrow 1006$ threes are there and 1006 twos are there.

$\therefore$ Sum $= 1006(2+3) = 1006 \times 5 = 5030$.

Answer: 5030.

8. In the figure $ABCD$ and $CEFG$ are squares of sides 6cm and 2cm respectively. The area of the shaded portion (in cm^2) is —— .



Solution:

Clearly $\angle CGE = \angle CEG = \angle BEH = \angle BHE = 45°$
$CG = CE = 2$cm
$\Rightarrow \quad BE = BH = 4$cm

Shaded area = Area of square $ABCD$+ Area of $\triangle GCE$

$-$ Area of $\triangle BHE = 6 \times 6 + \frac{1}{2} \times 2 \cdot 2 - \frac{1}{2} \cdot 4 \cdot 4$
$= 36 + 2 - 8 = 30 cm^2$

Answer: 30 cm^2

9. Master Ramanujan of Sixth standard was drawing squares of sides 1cm, 2 cm, 3 cm and so on. After doing this for sometime he added the areas of the squares he made. He got the sum of the areas as 1015 cm^2. The number of squares Ramanujan had drawn is —— .

Solution:

The area of square are $1^2, 2^2, 3^2, \cdots$, i.e. 1,4,9, $\cdots$.

The sum is given to be 1015. i.e., $1 + 4 + 9 + 16 + \cdots = 1015$

since for a primary level no formula is expected, we add numbers in a particular fashion.

$$(1+9)+(4+16)+25+(36+64)+(49+81)+100$$
$$+(121+169)+(144+196)$$
$$=10+20+25+100+130+100+290+340$$
$$=(10+290)+100+100+(20+130)+340+25$$
$$=650+340+25=1015\,.$$

So Ramanujan added 14 squares.

10. The tens digit of a four digit number is an even prime. The number is divisible by 5. The other digits are all prime numbers and all the digits are distinct. The sum of all such four digit numbers is —— .

Solution: Let the number be $abcd$.

Given that $C=2$.

Since the number is divisible by 5, $d=5$ or 0. The digits are distinct and all are primes the other two single digit primes are 3 and 7.

The numbers are 3725, 7325

Sum of the numbers

$$\begin{array}{r} 3725 \\ 7325 \\ \hline 11050 \\ \hline \end{array}$$

Answer: 11050.

SCREENING TEST- KAPREKAR CONTEST

(NMTC at SUB-JUNIOR LEVEL VII & VIII Standards)

PART - A

1. p is an odd prime. Then $\left(\frac{p+1}{2}\right)^2 - \left(\frac{p-1}{2}\right)^2$ is

(A) a fraction less than p .
(B) a fraction greater than p.
(C) a natural number not equal to p.
(D) a natural number equal to p.

Solution:

$$\left(\frac{p+1}{2}\right)^2 = \frac{(p+1)^2}{4} = \frac{p^2+2p+1}{4}$$

$$\left(\frac{p-1}{2}\right)^2 = \frac{(p-1)^2}{4} = \frac{p^2-2p+1}{4}$$

$$\left(\frac{p+1}{2}\right)^2 - \left(\frac{p-1}{2}\right)^2 = \frac{(p^2+2p+1)-(p^2-2p+1)}{4}$$

$$= \frac{4p}{4} = p.$$

Answer: (D)

2. $PQRS$ is a parallelogram. MP and NP divide $\angle SPQ$ into three equal parts ($\angle MPQ > \angle NPQ$) and MQ and NQ divide $\angle RQP$ into 3 equal parts ($\angle MQP > NQP$). If $k(\angle PNQ) = (\angle PMQ)$ then $k =$

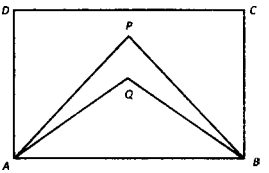
(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) $\frac{1}{3}$

Solution:

By hypothesis $\angle NPQ = \angle MPN = \angle SPM = x$ (say).

Similarly $\angle NQP = \angle MQN = \angle RQM = y$ (say).

Now $3x + 3y = 180^\circ \Rightarrow x + y = 60^\circ$



$$\begin{aligned} \angle PNQ &= 180^\circ - (x + y) = 120^\circ \\ \angle PMQ &= 180^\circ - (2x + 2y) = 180^\circ - 2(x + y) \\ &= 180^\circ - 2 \cdot 60 = 60^\circ \end{aligned}$$

Therefore $k(\angle PNQ) = \angle PMQ$ implies $k = \frac{1}{2}$.

Answer: (A)

3. In a foot ball league, a particular team played 50 games in a season. The team never lost three games consecutively and never won four games consecutively, in that season. If N is the number of games the team won in that season, then N satisfies

(A) $25 \leq N \leq 30$ (B) $25 \leq N \leq 36$
(C) $16 \leq N \leq 38$ (D) $16 \leq N \leq 30$

Solution: In order to get minimum of wins, the results of the team should be as follows:

$$\underbrace{LLW}_{\text{1st set}} \quad \underbrace{LLW}_{\text{2nd set}} \quad \underbrace{LLW}_{\text{3rd set}} \quad \underbrace{LLW}_{\text{4th set}} \quad \cdots\cdots \quad \underbrace{LLW}_{\text{16th set}} \quad LL$$

$\therefore$ The team can win a minimum of $16 \times 1 = 16$ games

$\therefore$ The team can win a minimum of 16 games and a maximum of 38 games. As the team win N games we have $16 \leq N \leq 38$.

Answer: (C).

4. a, b, c, d, e are five integers such that $a + b = b + c = c + d = d + e = 2012$, $a + b + c + d + e = 5024$. Then the value of $(d - a)$ is

(A) 8 (B) 12 (C) 10 (D) 4

Solution:

$$a+b=b+c \Rightarrow a=c$$
$$c+d=d+e \Rightarrow c=e$$
$$b+c=c+d \Rightarrow b=d$$

$a+b=b+c=c+d=d+e=2012$

$a+b+c+d+e=5024$

$\Rightarrow a+b+a+b+a=5024$

$\Rightarrow 3a+2b=5024$, or $2(a+b)+a=5024$

$\Rightarrow 2(2012)+a=5024 \Rightarrow a=1000$

$b=2012-a=1012, \quad d=b=1012$

$d-a=1012-1000=12.$

Answer: (B).

5. The last two digits of 3^{2012}, when represented in decimal notation, will be

(A) 81 (B) 01 (C) 41 (D) 21

Solution:

....81	61	41	21	01
×...81	×...81	×...81	×...81	×...81
81	61	41	21	01
8	8	8	8	8
61	41	21	01	81

From these observations, we can establish the following properties:

Property	A Num ending with digits	Mult. with a Num ending with digits	results in a Num ending with digits
(a)	81	81	61
(b)	61	81	41
(c)	41	81	21
(d)	21	81	01
(e)	01	81	81

$\therefore$ We have $3^4, 3^{24}, 3^{44}, 3^{64}, \cdots$, ending with digits 81.

$3^8, 3^{28}, 3^{48}, \cdots$, ends with digits 61.

$3^{12}, 3^{32}, 3^{52}, \cdots$, ends with digits 41.

$3^{16}, 3^{36}, 3^{56}, \cdots$, ends with digits 21.

$3^{20}, 3^{40}, 3^{60}, \cdots$, ends with digits 01.

$\therefore$ $3^{2012} = 3^{12+100(20)}$ ends with digits 41.

Answer: (C).

6. There exists positive integers x, y such that both the expressions $(3x + 2y)$ and $(4x - 3y)$ are exactly divisible by

(A) 11 (B) 7 (C) 23 (D) 17

Solution: $4(3x+2y)-3(4x-3y) = 12x+8y-12x+9y = 17y$.

Also $3(3x + 2y) + 2(4x - 3y) = 9x + 6y + 8x - 6y = 17x$.

$\therefore$ If 17 divides one among $(3x + 2y), (4x - 3y)$ it divides the other too. Hence these exists positive integers x, y such that both, $(3x + 2y), (4x - 3y)$ are exactly divisible by 17.

Answer: (D).

7. The years of 20^{th} century and 21^{st} century are of 4 digits. The number of years which are divisible by the product of the four digits of the year is

(A) 7 (B) 8 (C) 9 (D) none of these

Solution: For the 20^{th} century the years are from 1900 to 1999 and hence the product of the digits of the years have 9 as a factor. So it is enough if we consider only $\frac{1999-1900}{9} = \frac{99}{9} = 11$ numbers.

Of these 1900 to 1910 have zero as one of the digit. So it is enough to consider numbers from $(1911 - 1919)(1921 - 1929)\cdots(1991 - 1999)$.

1917, 1926, 1935, 1944, 1953, 1962, 1971, 1989, 1998 are the numbers divisible by 9.

(1) $1917 = 9 \times 213 = 9 \times 3 \times 71 = 27 \times 71$

Product of the digits is $1 \times 9 \times 1 \times 7 = 63$. 1917 is not divisible by 7.

(2) $1926 = 9 \times 214 = 9 \times 2 \times 107$.

Product of the digits is $1 \times 9 \times 2 \times 6 = 27 \times 4$ and the number 1926 is not divisible by 4 (neither by 27).

(3) $1935 = 9 \times 215 = 9 \times 5 \times 43$.

Product of the digits is $1 \times 9 \times 3 \times 5 = 27 \times 5$. 1935 is not divisible by 27.

(4) $1944 = 9 \times 216 = 9 \times 8 \times 27$.

Product of the digits is $1 \times 9 \times 4 \times 4 = 9 \times 16$. 16 does not divide 1944.

(5) $1953 = 9 \times 217 = 9 \times 7 \times 31$.

The product of digits is $1 \times 9 \times 5 \times 3 = 27 \times 5$. 1953 is not divisible by 5.

(6) $1962 = 9 \times 218 = 9 \times 2 \times 109$.

Product of the digits is $1 \times 9 \times 3 \times 4$. 1962 is not divisible by 4.

(7) $1971 = 9 \times 219 = 27 \times 73$ and the product of the digits is $1 \times 9 \times 7 \times 1$. The year 1971 is not divisible by 7.

(8) $1989 = 9 \times 221 = 9 \times 13 \times 17$.

The product of digits $1 \times 9 \times 8 \times 9$. The year 1989 is not divisible by product of digits.

(9) $1998 = 9 \times 222 = 9 \times 6 \times 37$.

The product of digits $1 \times 9 \times 9 \times 8$. The year 1998 is not divisible by $1 \times 9 \times 9 \times 8$.

For 21^{st} century all the year numbers from 2000 to 2099 contain one zero and divisibility by zero is not to be considered and hence no year number is divisible by the product of digits of the year number.

(Zero does not divide any number but every non zero number divides zero!).

8. If $x + \frac{1}{x} = -1$, then the value of $x^3 - \frac{1}{x^3}$ is

(A) 0 (B) 1 (C) -1 (D) 2

Solution:

$x + \frac{1}{x} = -1$ given $\Rightarrow x^2 + 1 = -x$

$\Rightarrow x^2 = -x - 1 = -x + (x + \frac{1}{x}) = \frac{1}{x}$

$\Rightarrow x^3 = 1$ and $x^3 - \frac{1}{x^3} = 0$.

Answer: (A).

9. The number of natural numbers $a(< 100)$ such that $(a^3 - a^2)$ is a square of a natural number is

(A) 7 (B) 8 (C) 9 (D) 10

Solution: $a^3 - a^2 = a^2(a - 1)$. For $a - 1$ to be a square number, a should be $= 2, 5, 10, 17, 26, 37, 65, 82$. There are 9 such numbers.

Answer (C).

10. Let n be the smallest nonprime integer greater than 1 with no prime factor less than 10. Then

(A) $100 < n \leq 110$ (B) $110 \leq n \leq 120$
(C) $120 < n \leq 130$ (D) $130 < n \leq 140$

Solution: Prime factors are not less than 10.
$\Rightarrow$ factors are 10,11,12,13, $\cdots$. Least prime factor is 11. Possible value of n are 110, 120, 121, 144. Correct number non prime is 121. i.e., $120 < 121 \leq 130$.
Answer: (C).

11. A point P inside a rectangle $ABCD$ is joined to the angular points. Then

(A) Sum of the areas of two of the triangles so formed is equal to the sum of the other two.

(B) The sum of the areas of the triangles so formed is a whole number whatever may be the dimensions of the rectangle.

(C) The sum of the areas of a pair of opposite triangles is greater than half the area of the rectangle.

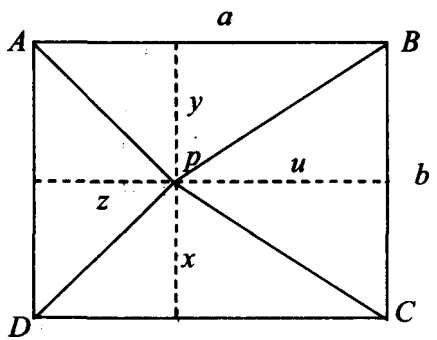
(D) None of these

Solution:

Consider triangles APB and CPD.

Sum of areas of $\triangle APB$ and $\triangle CPD$

$$= \frac{1}{2}a \cdot y + \frac{1}{2}a \cdot x$$
$$= \frac{1}{2}a(x+y)$$
$$= \frac{1}{2}ab.$$



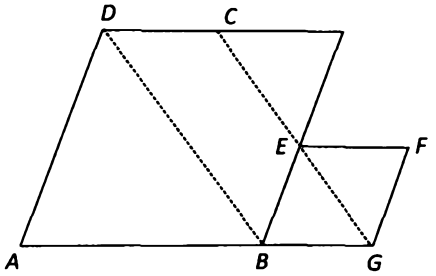
Consider triangles APD and BPC.

Sum of areas of $\triangle APD$ and $\triangle BPC$

$$\begin{aligned} &= \frac{1}{2}b(z) + \frac{1}{2}b(a) \\ &= \frac{1}{2}b(z+b) \\ &= \frac{1}{2}b \cdot a \\ &= \frac{1}{2}ab. \end{aligned}$$

Answer: (A).

12. In the adjoining figure $ABCD$ and $BGFE$ are rhombuses. $AB = 10\,\text{cm}$, $GF = 3\,\text{cm}$. GE meets DC at H. $\angle A = 60°$.



The perimeter of $ABEHD$ is (in cm)

(A) 47 (B) 40 (C) 39 (D) 33

Solution: $BGFE$ is a rhombus and $\angle EBG = 60°$.

$BE = GF = 3\,\text{cm}$. So $EC = BC - BE = AB - BE = 10 - 3 = 7\,\text{cm}$. $ABCD$ is a rhombus implies $\angle A = \angle C = 60°$. Noting that $\triangle BGE$ is equilateral, $\angle HEC = 60°$ (vertically opposite angle). $\therefore$ $\triangle HCE$ is equilateral and we have $CH = HE = CE = 7\text{cm}$. The required perimeter is $AB + BE + EH + HD + DA = 10 + 3 + 7 + 3 + 10 = 33$.

Answer: (D).

13. If $3a + 1 = 2b - 1 = 5c + 3 = 7d + 1 = 15$, then the value of $(3a - b + 5c - 9d)$ is

(A) 0 (B) 1 (C) 2 (D) 3

Solution:

$$\begin{aligned} 3a+1=15 &\Rightarrow 3a=14 \\ 2b-1=15 &\Rightarrow 2b=16, b=8 \\ 5c+3=15 &\Rightarrow 5c=12 \\ 7d+1=15 &\Rightarrow 7d=14, d=2 \end{aligned}$$

$3a-b+5c-9d=14-8+12-18=0$.

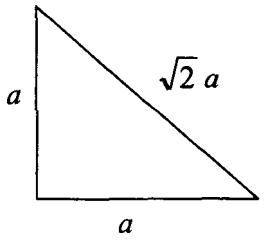
Answer: (A)

14. The perimeter of an isosceles right angled triangle is 2012. Its area is

(A) $2012(3-\sqrt{2})$ (B) $(1006)^2(3-\sqrt{2})$
(C) $(2012)^2$ (D) $(1006)^2$

Solution:

Perimeter of isosceles right angled triangle is $\Big\}2a+\sqrt{2}a.$



Given

$$\begin{aligned} 2a+\sqrt{2}a &= 2012 \\ a &= 2012/(2+\sqrt{2}) = \frac{2012(2-\sqrt{2})}{2} = 1006(2-\sqrt{2}) \\ \text{Area } \frac{a^2}{2} &= \frac{1006^2(2-\sqrt{2})^2}{2} = \frac{1006^2(4-4\sqrt{2}+2)}{2} \\ &= \frac{1006^2(6-4\sqrt{2})}{2} = \frac{1006^2\times 2\times(3-2\sqrt{2})}{2} \end{aligned}$$

Answer: (B).

15. Five two digit numbers (none of the digits is zero) add up to 100. If each digit is replaced by its 9 complement, then the sum of these five new numbers is

(A) 295 (B) 195 (C) 380 (D) 395

Solution: Let the five numbers be $a_1b_1, a_2b_2, a_3b_3, a_4b_4$ and a_5b_5. Where a_is are the 10^{th} place and b_is are units place

then we get,

$$(10a_1+b_1)+(10a_2+b_2)+(10a_3+b_3)+(10a_4+b_4)+(10a_5+b_5) = 100$$

$$10(a_1 + a_2 + a_3 + a_4 + a_5) + b_1 + b_2 + b_3 + b_4 + b_5 = 100 \quad (1)$$

Writing the above numbers changing each of digits to its 9 complement we get,

$$\begin{aligned}
&[10(9-a_1)+(9-b_1)]+[10(9-a_2)+(9-b_2)]+[10(9-a_3)+(9-b_3)] \\
&\quad +[10(9-a_4)+(9-b_4)]+[10(9-a_5)+(9-b_5)] \\
&= 10[45-(a_1+a_2+a_3+a_4+a_5)] \\
&\quad +45-(b_1+b_2+b_3+b_4+b_5) \\
&= 450+45-[10(a_1+a_2+a_3+a_4+a_5) \\
&\quad +(b_1+b_2+b_3+b_4+b_5)] \\
&= 495-100 \text{ (using (1))} \\
&= 395.
\end{aligned}$$

Answer: (D).

Note: Let the numbers be 18,19,20,21,22 whose sum is 100.

If each digit is replace by its 9 compliment

$$\begin{array}{rcl}
18 & \rightarrow & 81 \\
19 & \rightarrow & 80 \\
20 & \rightarrow & 79 \\
21 & \rightarrow & 78 \\
22 & \rightarrow & 77 \\
\hline
100 & \rightarrow & 395 \\
\hline
\end{array}$$

The sum of all 9 compliment numbers is 395.

Note: This example does not go with the hypothesis since one of the numbers (20) has a digit zero.

PART - B

1. The sum of two natural numbers is 484. Their HCF is 11. The number of such possible pairs is ——

   Solution: Since the HCF of two natural numbers is 11, both the numbers are multiples of 11.

   Let the numbers be $11a$ and $11b$ where a, b are co-prime.

   $\therefore\ 11a + 11b = 484 \Rightarrow 11(a+b) = 11 \times 44$

   $a + b = 44$ where HCF of a, b is 1.

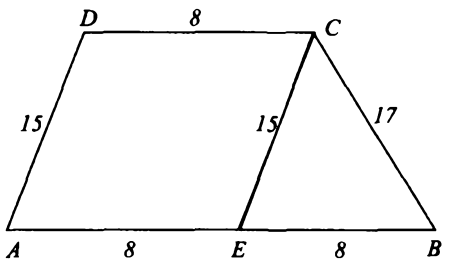
   The possible pairs of (a, b) are, (1,43) (3,41) (5,39) (7,37) (9,35) (13,31) (15,29) (17,27) (19,25) and (21,23).

   $\therefore$ The number of possible such pairs is 10.

   Answer: 10.

2. $ABCD$ is a trapezium with AB and CD parallel. If $AB = 16\,\text{cm}$, $BC = 17\,\text{cm}$, $CD = 8\,\text{cm}$, $DA = 15\,\text{cm}$ then the area of the trapezium (in cm^2) is ——

   Solution:

   Draw $CE \| AD$, as shown.

   We get $AECD$, a parallelogram.

   $\therefore\ AE = CD = 8\,\text{cm}$.

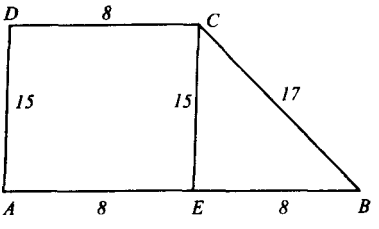
   $BE = AB - AE = (16 - 8)\,\text{cm}$

   $BE = 8\,\text{cm}$

   Consider $\triangle BEC$

   $$\text{we have } BE^2 + EC^2 = BC^2$$
   $$(8^2 + 15^2) = 17^2$$

   $\therefore\ BEC$ is a right angled $\triangle$ $90°$ at E.



$\therefore$ Area of trapezium
= Area of rectangle
$AECD$+ Area of triangle
$BEC = (15 \times 8 + \frac{1}{2} \times 15 \times 8)$
$= 180\text{cm}^2$.

3. The number of multiples of 9 less than 2012 and having sum of the digits as 18 is ——

Solution: The only 2 digit number with sum of the digits 18 is 99.

Let us tabulate 3 digit numbers with sum of the digits 18 as follows:

End digit	Numbers	How many?
9	189, 279, 369, $\cdots$, 909	9
8	198,288,378, $\cdots$,918	9
7	297, 387,477, $\cdots$, 927	8
6	396, 486,576, $\cdots$, 936	7
5	495,585,675, $\cdots$, 945	6
4	594,684,774, $\cdots$, 954	5
3	693 ,783,873, $\cdots$, 963	4
2	792 ,882,972	3
1	891 ,981	2
0	990	1
		54

Note that the last 2 digits of these 54 3-digit numbers are different from each other.

To get 4-digit numbers < 2012 with the sum of the digits 18, we can replace the starting digits of each 3-digit number (in the 54 numbers we got) as follows:

Starting digit	Replace by
1	10
2	11
3	12
4	13
5	14
6	15
7	16
8	17
9	18

There is no four digit number from 2000 to 2011 whose sum of the digits is 18.

$\therefore$ The number of multiples of 9, less than 2012 and having sum of the digits as 18 is $1 + 54 + 54 = 109$.

4. AB, CD, EF, GH and IJ are five 2-digit numbers such that each letter stands for a different digit. The largest possible sum of these five numbers is ——

Solution: To get the largest sum the tens placed digits must be of higher values i.e., A, C, E, G, I should be as large as possible. But they are different.

$\therefore$ The values of A, C, E, G, I should be 9,8,7,6,5 in some order.

Here are few possibilities:

	(1)	(2)	(3)
AB	93	84	61
+CD	+84	+73	+54
+EF	+71	+62	+93
+GH	+60	+91	+80
+IJ	+52	+50	+72
	360	360	360

The largest possible sum of these five numbers is 360.

5. Two consecutive natural numbers are respectively divisible by 4 and 7. The sum of their respective quotient is 8. Then the sum between the numbers is ——

Solution: Let the numbers be $n, n+1$

$$\frac{n}{4} = a \quad \text{and} \quad \frac{n+1}{7} = b.$$

$$n = 4a \quad \Rightarrow \quad 4a + 1 = 7b$$

$$4a - 7b = -1 \qquad (1)$$

$$\text{Also } a + b = 8 \quad \Rightarrow \quad 4a + 4b = 32 \qquad (2)$$

$$\begin{aligned} \text{Solving (1) and (2)} \quad 11b &= 33 \\ b &= 3 \\ \Rightarrow \quad n &= 20 \\ n+1 &= 21 \end{aligned}$$

$\therefore$ Sum of numbers $= 41$.

6. p is the difference between a real number and its reciprocal. q is the difference between the square of the same real number and the square of the reciprocal. Then the value of $p^4+q^2+4p^2$ is ——

Solution:

$$p = x - \frac{1}{x} \text{ and } q = x^2 - \frac{1}{x^2}$$

$$p^4 = (x - \frac{1}{x})^4 = x^4 - 4x^2 + 6 - \frac{4}{x^2} + \frac{1}{x^4}$$

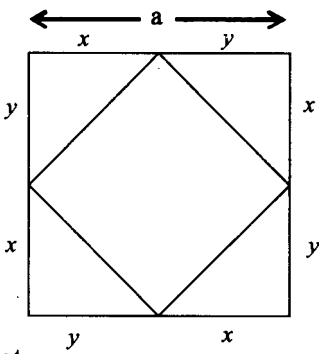
$$\begin{aligned} p^4 + q^2 + 4p^2 &= x^4 + \frac{1}{x^4} - 4x^2 - \frac{4}{x^2} + 6 + x^4 + \frac{1}{x^4} \\ &\quad -2 + 4(x^2 + \frac{1}{x^2} - 2) \\ &= 2(x^4 + \frac{1}{x^4}) - 4 \\ &= 2(x^4 + \frac{1}{x^4} - 2) = 2q^2. \end{aligned}$$

Answer: $2q^2$.

7. A square is inscribed in another square each of whose four vertices lies on each side of the square. The area of the smaller square is $\frac{25}{49}$ times the area of the bigger one. Then the ratio with which each vertex of the smaller square divides the side of the bigger square is ——

Solution: Let $x + y = a$. By data,

$$x^2 + y^2 = \frac{25}{49}(x+y)^2$$
$$49x^2 + 49y^2 = 25x^2 + 25y^2 + 50xy$$
$$24x^2 + 24y^2 = 50xy$$
$$12x^2 - 25xy + 12y^2 = 0$$
$$12\frac{x^2}{y^2} - 25\frac{x}{y} + 12 = 0$$
$$\frac{x}{y} = \frac{25 \pm \sqrt{625 - 576}}{2.12}$$
$$= \frac{25 \pm 7}{24} = \frac{32}{24} = \frac{4}{3}.$$



Justification for $+$. Since x cannot be equal to y. We can assume $x > y$.

Answer: $\frac{4}{3}$.

8. a, b, c are three positive numbers. The second number is greater than the first by the amount the third number is greater than the second. The product of the two smaller numbers is 85 and that of the two bigger numbers is 115. Then the value of $(2012a - 1006c)$ is ——

Solution:

$$b - a = x$$
$$c - b = x$$
$$b - a = c - b$$
$$2b = a + c$$
$$b = \frac{a+c}{2}$$

a, b, c are in A.P.

$ab = 85$ and $bc = 115$

$b(a + c) = 200 \Rightarrow b = 10$.

We have $a = 8.5$, $b = 10$, $c = 11.5$.

$$\begin{aligned} 2012a - 1006c &= 1006(2(8.5) - 11.5) \\ &= 1006(17 - 11.5) \\ &= 1006 \times 5.5 \\ &= 5533. \end{aligned}$$

Answer: 5533.

9. If $x = \dfrac{y}{y+1}$ and $y = \frac{a-2}{2}$ the value of $x(y+2) + \dfrac{x}{y} + \dfrac{y}{x}$ when $a = 2012$ is ——

Solution: $x = \frac{y}{y+1}$, $y = \frac{a-2}{2}$

$$\begin{aligned} y + 1 = \frac{a-2}{2} + 1 &= \frac{a}{2} \\ y + 2 = \frac{a-2}{2} + 2 &= \frac{a+2}{2} \\ \frac{y}{y+1} = \frac{a - 2/2}{a/2} &= \frac{a-2}{a} = x \\ \frac{x}{y} = \frac{a - 2/a}{a - 2/2} &= \frac{2}{a} \Rightarrow \frac{y}{x} = \frac{a}{2} \\ x(y+2) + \frac{x}{y} + \frac{y}{x} &= \frac{a-2}{a}\left(\frac{a+2}{2}\right) + \frac{2}{a} + \frac{a}{2} \\ &= \frac{a^2 - 4 + 4 + a^2}{2a} = \frac{2a^2}{2a} = a. \end{aligned}$$

Answer: 2012.

10. PSR is an isosceles triangle in which $PS = PR$. SP is produced to O such that $PO = SP$. Then $\angle SRO$ is equal to ——

Solution:

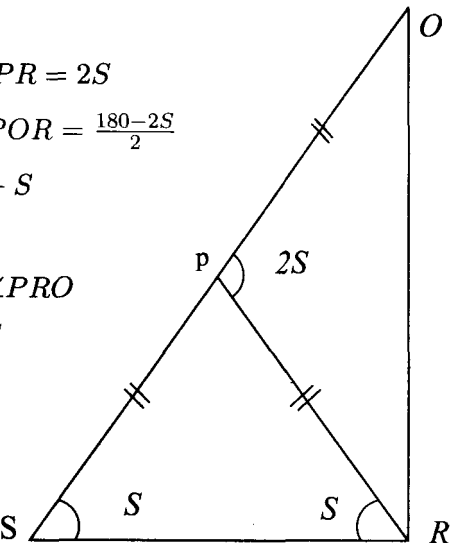
$PS = PR \Rightarrow \angle S = \angle R$.

In $\triangle PSR$ External $\angle SPR = 2S$

In $\triangle OPR$ $\angle PRO = \angle POR = \frac{180-2S}{2}$

$\angle PRO = 90 - S$

$$\begin{aligned} \angle SRO &= \angle SRP + \angle PRO \\ &= S + 90 - S \\ &= 90^\circ \\ \angle SRO &= 90^\circ. \end{aligned}$$



Answer: 90°.

Aliter: With the hypothesis with P as centre draw a circle of radius PS. The circle will pass through O and R and OPS is the diameter. We know that angle in a semicircle is 90°. Hence $\angle SRO = 90^\circ$.

11. a, b, c, d are five integers such that $a + b = b + c = c + d = d + e = 2012$ and $a + b + c + d + e = 5024$. Then the value of $(d - a) =$ ——

Solution: Refer Question 4 of Part-A.

12. A class contains three girls and four boys. Every Saturday, five students go on a picnic, a different group is sent each week. During the picnic, each person (boy or girl) is given a Cake by the accompanying teacher. After all possible groups of five have gone once; the total number of cakes received by the girls during the picnic is ——

Solution:

Assume A, B, C are girls. D, E, F, G are boys. Let us tabulate as follows:

CASE	No. of Girls in 5 persons	No of Boys in 5 persons	Possibilities of Girls	Possibilities of Boys
1	3	2	ABC	DE or DF or DG or EF or EG or FG
2	2	3	AB or AC or BC	DEF or DEG or DFG or EFG
3	1	4	A or B or C	DEFG

The number of times A (girl) went to picnic

$$\underset{\text{Case 1}}{6} \quad + \underset{\text{Case II}}{8} \quad + \underset{\text{Case III}}{1} \quad = 15$$

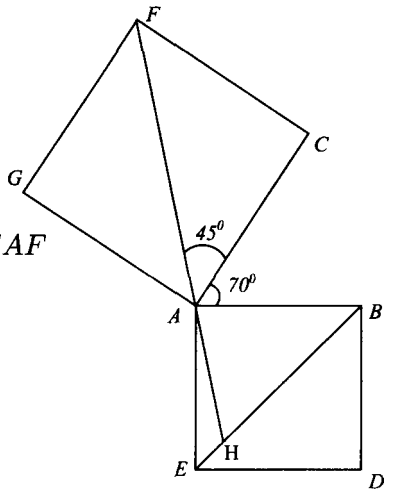
Each girl would have gone 15 times for the picnic.

$\therefore$ To total number of cakes received by girls together
$= 3 \times 15 = 45$.

Answer: 45.

13. CAB is an angle whose measure is $70°$. $ACFG$ and $ABDE$ are squares drawn outside the angle. The diagonal FA meets BE at H. Then the measure of the angle EAH is ——

Solution:



$\angle CAB = 70°$

$\angle CAF = 45°$

FAH is straight line

$\angle BAH = 180° - \angle CAB - CAF$

$= 180° - 70° - 45°$

$= 65°$

$\angle BAE = 90°$

$\angle EAH = 90° - BAH$

$= 90° - 65° = 25°$

Answer: $25°$.

14. The number of prime numbers p for which $p+2$ a
are also prime numbers is ——

Solution: $p+5$ is a prime which has to be odd

$\Rightarrow$ p can not be odd $\Rightarrow$ $p=2$.

But for $p=2$, $p+2=4$ is not a prime.

$\therefore$ Number of primes for which $p+2, p+5$ are also pr
zero.

15. The number of integer pairs (m,n) which satisfies $m(n^2+$
48 is ——

Solution:

m	n^2+1	n^2	(n is an integer)
1	48	47	not possible
2	24	23	not possible
3	16	15	not possible
4	12	11	not possible
6	8	7	not possible
8	6	5	not possible
12	4	3	not possible
16	3	2	not possible
24	2	1	possible value
48	1	0	possible value

Integer pairs are $(24,1)$ $(48,0)$.

SCREENING TEST- BHASKARA CONTEST
(NMTC at JUNIOR LEVEL IX & X Standards)

PART - A

1. Two regular polygons of same number of sides have sides 40cm and 9cm in length. The lengths of the side of another regular polygon of the same number of sides and whose area is equal to the sum of the areas of the given polygons is (in cm)

   (A) 49 (B) 31 (C) 41 (D) 360

   Solution: Area of regular polygon $= Ka^2$ where 'a' is the length of a side and K depends on the number of sides.

   From the data we have,

$$\begin{aligned} K(40^2+9^2) &= K(1600+81) = KC^2 \\ &\Rightarrow C = 41. \end{aligned}$$

   Answer: (C).

2. If $x > y > 0$ and $\frac{x+y}{x-y} = \sqrt{2}$, the value of $\frac{x^2+y^2}{xy}$ is

   (A) 5 (B) 4 (C) 1 (D) 6

   Solution:

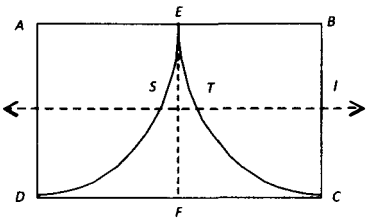
$$\frac{x+y}{x-y} = \sqrt{2} \Rightarrow \frac{\sqrt{2}}{1} = \frac{x+y}{x-y}$$

   By componendo and dividendo we get,

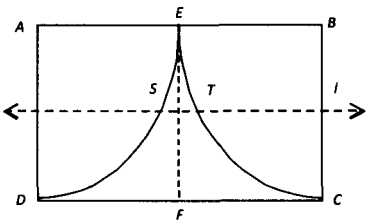
$$\begin{aligned} \frac{\sqrt{2}+1}{\sqrt{2}-1} &= \frac{2x}{2y} = \frac{x}{y} \\ \Rightarrow \quad \frac{x}{y} &= \frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)(\sqrt{2}+1)} = 3+2\sqrt{2} \\ \frac{y}{x} &= \frac{1}{3+2\sqrt{2}} = 3-2\sqrt{2} \\ \frac{x^2+y^2}{xy} &= \frac{x}{y}+\frac{y}{x} = 3+2\sqrt{2}+3-2\sqrt{2} = 6. \end{aligned}$$

   Answer: (D).

3. In rectangle $ABCD, AB = 2BC = 4\,\text{cm}$ E and F are midpoints of AB and CD respectively. ESD and ETC are arcs of circles centred at A and B respectively. If the perpendicular bisector line l of EF cuts the arcs at S and T as in the diagram, then ST is equal to (in cm)



(A) $(4-2\sqrt{3})$ (B) $(3+\sqrt{3})$
(C) $(2+2\sqrt{3})$ (D) $(4\sqrt{3}-2)$



Solution: Let the perpendicular bisector of EF cut AD and BC at G, H respectively G. Clearly $\triangle AGS$, $\triangle BTH$, are congruent right angled triangles

$$\begin{aligned} AS &= BT = \text{radius} = 2\text{cm} \\ AG &= BH = \frac{1}{2}AD = 1\text{cm} \\ \angle AGS &= \angle BHT = 90^\circ \\ GS &= TH = \sqrt{2^2 - 1^2} \\ \Rightarrow \quad GS &= TH = \sqrt{3}\text{cm} \\ ST &= GH - (GS + TH) \\ &= (4 - 2\sqrt{3}). \end{aligned}$$

Answer: (A).

4. The value of 'a' for which the expression

$$\left\{\left(a^{\frac{1}{4}} - a^{\frac{1}{8}} + 1\right)^{-1} + \left(a^{\frac{1}{4}} + a^{\frac{1}{8}} + 1\right)^{-1} - \frac{2\sqrt[4]{a} - 2}{\sqrt{a} - \sqrt[4]{a} + 1}\right\}^{-1}$$

$-2^{\log_4 a - 2}$ takes the value 2012 is

(A) 4048 (B) 6036 (C) 6037 (D) 4047

Solution: Simplifying the expression we get that the given expression $= \frac{a+1}{4}$.

As per hypothesis it takes the value 2012

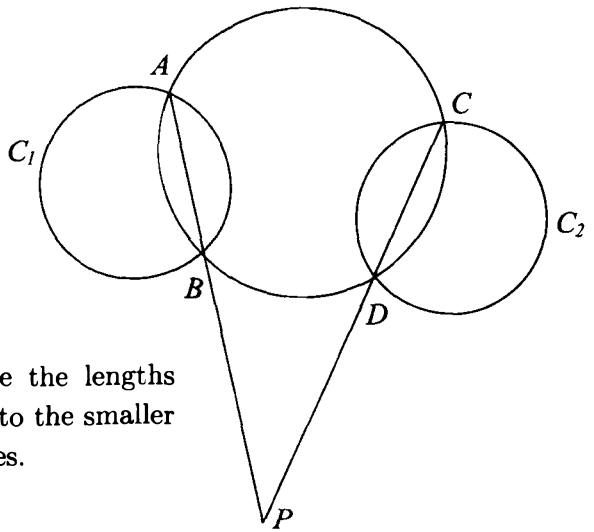
$$\begin{aligned} \Rightarrow \quad \frac{a+1}{4} &= 2012 \\ a+1 &= 8048 \\ a &= 8047 \end{aligned}$$

Answer: (D).

5. C_1 and C_2 are two non-intersecting circles whose radii are in the ratio 1:2. A third circle cuts the smaller circle at A and B and the bigger one at C and D. AB and CD intersect at P. The ratio of the lengths of the tangents form P to the circles C_1 and C_2 is

(A) 1:4 (B) 1:8 (C) 1:1 (D) 1:2

Solution:



$PA \cdot PB = t_1^2$

$PC \cdot PD = t_2^2$

where t_1, t_2 are the lengths of the tangents to the smaller and bigger circles.

$t_1 : t_2 = 1 : 1$.

Answer: (C). Here the ratio of the radii of C_1, C_2 is immaterial provided they are unequal.

6. a, b, c are real numbers and none of them zero and $E = \left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 + \left(ab+\frac{1}{ab}\right)^2 - \left(a+\frac{1}{a}\right)\left(b+\frac{1}{ab}\right)\left(ab+\frac{1}{ab}\right)^2$. Then E is equal to

(A) 2012 when $a = b = 2012$ (B) 2012 when $ab = 2012$
(C) 4 for all real values of a and b
(D) 2012 for all real values of a and b

Solution:

$$\begin{aligned}\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(ab+\frac{1}{ab}\right) &= \left(ab+\frac{1}{ab}+\frac{a}{b}+\frac{b}{a}\right)\left(ab+\frac{1}{ab}\right)\\ &= \left(ab+\frac{1}{ab}\right)^2+\left(\frac{a}{b}+\frac{b}{a}\right)\left(ab+\frac{1}{ab}\right)\\ &= \left(ab+\frac{1}{ab}\right)^2+\left(a^2+\frac{1}{a^2}+b^2+\frac{1}{b^2}\right)\end{aligned}$$

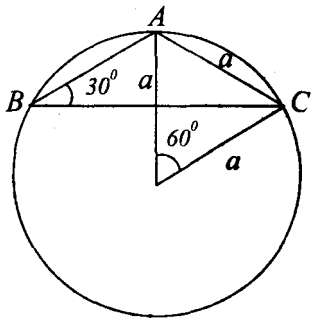
$$E = a^2+\frac{1}{a^2}+b^2+\frac{1}{b^2}+4+\left(ab+\frac{1}{ab}\right)^2 - \left[\left(ab+\frac{1}{ab}\right)^2+a^2+\frac{1}{a^2}+b^2+\frac{1}{b^2}\right] = 4$$

Answer: (C).

7. The angles of a triangle are in the ratio 2:3:7. The length of the smallest side is 2012cm. The radius of the circum circle of the triangle (in cm) is

(A) 2013 (B) 2011 (C) 4024 (D) 2012

Solution:



We have $2\theta + 3\theta + 7\theta = 180^\circ$

$12\theta = 180 \Rightarrow \theta = 15^\circ$

The angles of triangle are $30^\circ, 45^\circ$ and 105° smallest side

$AC = a$ since $\angle AOC = 2(30^\circ) = 60^\circ$

$a =$ radius $= 2012\,\text{cm}.$

Answer: (D)

8. If $a = 2012, b = -1005, c = -1007$, then the value of $\dfrac{a^4}{b+c} + \dfrac{b^4}{c+a} + \dfrac{c^4}{a+b} + 3abc$ is

(A) 2012 (B) 1 (C) 0 (D) $(2012)^3$

Solution: We have $a + b + c = 2012 - 1005 - 1007 = 0$

$\Rightarrow \quad a^3 + b^3 + c^3 = 3abc.$

Given expression $= -a^3 - b^3 - c^3 + 3abc = 0.$

Answer: (C)

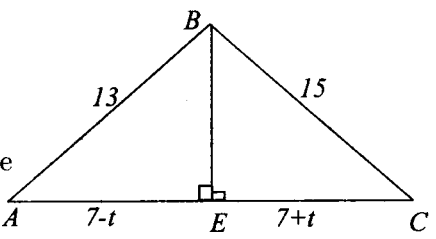
9. ABC is a triangle with $AB = 13\,\text{cm}$, $BC = 14\,\text{cm}$, and $CA = 15\,\text{cm}$. AD and BE are the altitudes from A and B to BC and AC respectively. H is the point of intersection of AD and BE. Then the ratio $\frac{HD}{HB} =$

(A) $\frac{3}{5}$ (B) $\frac{12}{13}$ (C) $\frac{4}{5}$ (D) $\frac{5}{9}$

Solution:

Let BE be the altitude
$AC, AS \quad AC = 14\,\text{cm}$

Let $EC = (7 + t)$ cm
$AE = (7 - t)$ cm
By Pythagoras theorem, we have



$$AB^2 - AE^2 = BE^2$$
$$BC^2 - EC^2 = BE^2$$

$\Rightarrow \quad 13^2 - (7-t)^2 = BE^2 = 15^2 - (7+t)^2$

$\Rightarrow \quad (7-t)^2 - (7-t)^2 = 15^2 - 13^2$

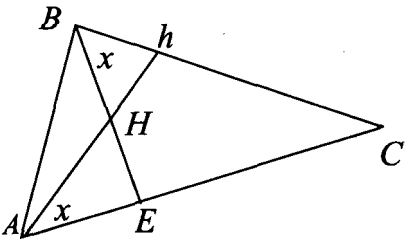
$\Rightarrow \quad 4 \times 7 \times t = 28 \times 2$

$\Rightarrow \quad t = 2$

$\therefore AE = 5\text{cm} \quad EC = 9\text{cm}.$

Consider $\triangle ADC$ and $\triangle BEC$.

$\angle ACD = \angle BCE$ (common)
$\angle ADC = \angle BEC = 90°$
$\therefore \quad \angle DAC = \angle EBC$
(Third angles are equal)



This is because the sum of the angles of any triangle is $180°$.

$\therefore \quad \triangle ADC$ and $\triangle BEC$

are similar triangles.

Now, consider $\triangle BHD$ and $\triangle BEC$

$\angle HBD = \angle EBC$ common;

$\angle BOH = \angle BEC = 90°$

$\therefore \quad \angle BHD = \angle BCE$ (Third angles are equal)

$\therefore \quad \triangle BHD$ and $\triangle BCE$ are similar triangles

$$\Rightarrow \quad \frac{HD}{HB} = \frac{CE}{CB} = \frac{9\text{cm}}{15\text{cm}} = \frac{3}{5}.$$

$\therefore$ Correct option is (A) $\frac{3}{5}$

Answer: (A).

10. For how many positive integrals values of $x \leq 100$ is $(3^x - x^2)$ divisible by 5?

(A) 16 (B) 20 (C) 24 (D) 36

Solution:

The decimal expansion of 3^x ends with 3,9,7 and 1 only.

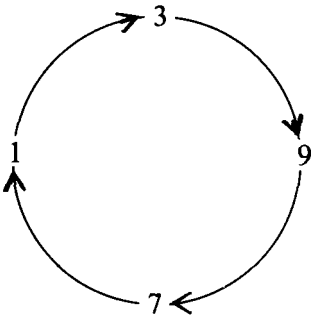
Here x stands for a natural number (+ve integer)

Character of x	3^x ends with digit	End digit cycle of 3^x
A multiple of 4+1	3	
A multiple of 4+2	9	
A multiple of 4+3	7	
A multiple of 4+0	1	

The decimal expansion of x^2 ends with digit 1,4,9,6,5,6,9,4,1,0 only.

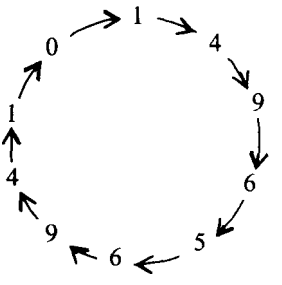
Here x stand for positive integer.

End digit of x	End digit of x^2	End digit cycle of x^2
1	1	
2	4	
3	9	
4	6	
5	5	
6	6	
7	9	
8	4	
9	1	
0	0	



As LCM of 4,10 is 20, the end digit of $(3^x - x^2)$ forms a cycle of 20 numbers.

Let us table as follows:



Positive integer x	End digit of 3^x	End digit of x^2	End digit of $3^x - x^2$
1	3	1	2
2	9	4	5
3	7	9	8
4	1	6	5
5	3	5	8
6	9	6	3
7	7	9	8
8	1	4	7
9	3	1	2
10	9	0	9
11	7	1	6
12	1	4	7
13	3	9	4
14	9	6	3
15	7	5	2
16	1	6	5
17	3	9	4
18	9	4	5
19	7	1	6
20	1	0	1
21	3	1	2
22	9	4	5

This cycle repeats.

$\therefore$ The number of positive integral values of $x \leq 100$ for which $3^x - x^2$ ends with digit '5' or divisible by 5 is 4×5 (cycle of 20) = **20**

Answer: (B).

11. If one root of $\sqrt{a-x}+\sqrt{b+x}=\sqrt{a}+\sqrt{b}$ is 2012, then a possible value of a, b is

(A) (2000, 2012) (B) (4024, 2012)

(C) (1000,1012) (D) (1012, 1000)

Solution: $\sqrt{a-x}+\sqrt{b+x}=\sqrt{a}+\sqrt{b}$.

Squaring both sides we get,

$$a-x+b+x+2\sqrt{(a-x)(b+x)}=a+b+2\sqrt{ab}$$

$$\Rightarrow \quad 2\sqrt{(a-x)(b+x)}=2\sqrt{ab}$$

$$(a-x)(b+x)=ab$$

$$ab-bx+ax-x^2=ab$$

$$\Rightarrow \quad x^2+bx-ax=0$$

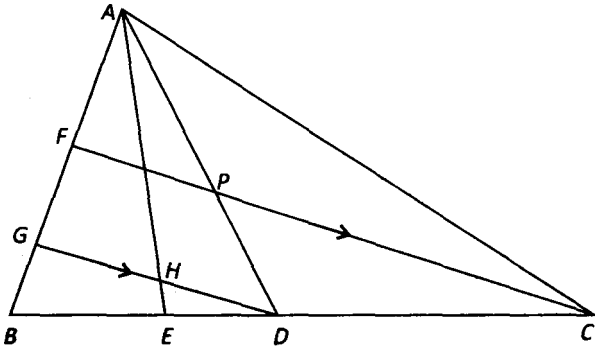
$$x(x+b-a)=0$$

$$\Rightarrow \quad x=0 \text{ or } x=a-b$$

$$\therefore \quad a-b=2012.$$

Answer: (B)

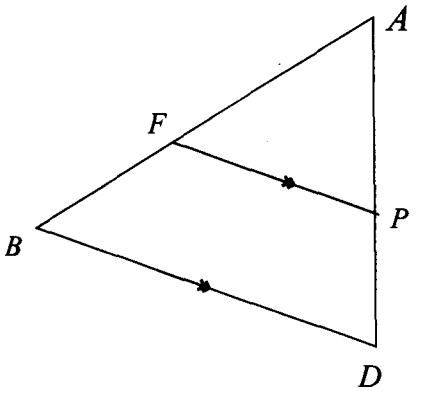
12. In the figure shown, $BD=CD$, $BE=DE$, $AP=PD$ and $DG||CF$.



Then $\frac{\text{area of } \triangle ADH}{\text{area of } \triangle ABC}$ is equal to

(A) $\frac{1}{5}$ (B) $\frac{2}{9}$ (C) $\frac{3}{13}$ (D) $\frac{1}{6}$

Solution: Let I be the point of intersection of CF and AE, as shown.



Consider $\triangle AGD$

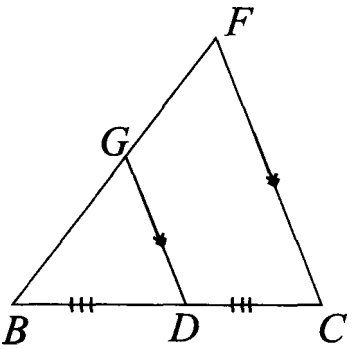
$FP||GD$ and p is mid point of AD.

$\therefore$ By converse of mid point theorem, F will mid point of AG.

$$\therefore \quad AF = FG \tag{1}$$

Now consider $\triangle FBC$

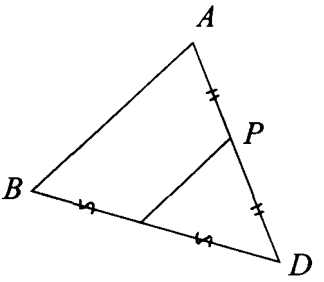
$GD||FC$ and D is mid point of BC.



$\therefore$ By converse of mid point theorem G will be mid point of FB.

$$\therefore \quad FG = GB \tag{2}$$

From (1) and (2), $AF = FG = GB$.



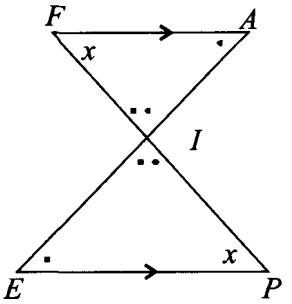
But $AF + FG + GB = AB$

$\Rightarrow AF = FG = GB = \frac{1}{3}AB$.

Consider $\triangle ABD$

By the mid point theorem $PE||AB$ and $PE = \frac{1}{2}AB$.

Now consider $\triangle AFI$ and $\triangle PIE$, as shown.



By alternate angles,

$\angle AFI = \angle IPE$ and

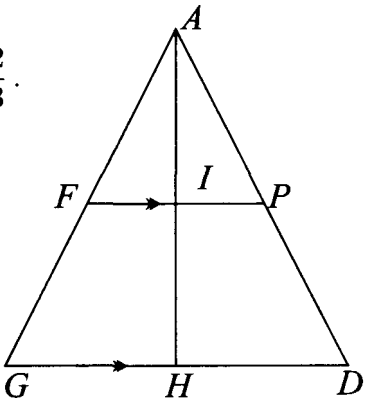
$\angle FAI = \angle IEP$

$\angle FIA = \angle PIE$ (vertically opp.)

$\therefore$ $\triangle AFI$ and $\triangle EPI$ are similar triangles.

$$\therefore \quad \frac{AF}{EP} = \frac{FI}{IP} = \frac{AI}{IE}$$

$$\therefore \quad \frac{AF}{EP} = \frac{FI}{IP} = \frac{IA}{IE} = \frac{\frac{1}{3}AB}{\frac{1}{2}AB} = \frac{2}{3}.$$



Now consider $\triangle AGD$

$FP||GD$.

$\therefore$ $FI||GH, IP||HD$

$$\frac{FI}{IP} = \frac{GH}{DH} = \frac{2}{3}.$$

$$\therefore \quad DH = \frac{3}{5}\,DG \text{ (Area of } \triangle ADH = \frac{3}{5} \text{ Area of } \triangle ADG) \quad (3)$$

Area of $\triangle BGD = \frac{1}{6}\times$ Area of $\triangle ABC$

since $GB = \frac{1}{3}\ AB$ and $BD = \frac{1}{2}BC$.

Area of $\triangle ABD = \frac{1}{2}\times$ Area of $\triangle ABC$ as $BD = DC = \frac{1}{2}\cdot BC$

$\therefore$ Area of $\triangle ADG =$ Area of $\triangle ABD -$

Area of $\triangle BGD = (\frac{1}{2} - \frac{1}{6}) \times$ Area of $\triangle ABC)$.

$\Rightarrow$ Area of $\triangle ADG = \frac{1}{3} \times$ Area of $\triangle ABC$ (4)

$\therefore$ Area of $\triangle ADH = \frac{3}{5} \times \frac{1}{3} \times$ Area of $\triangle ABC$ from (3) and (4)

$= \frac{1}{5}$ of Area of $\triangle ABC$. $\therefore \quad \frac{\text{Area of } \triangle ADH}{\text{Area of } \triangle ABC} = \frac{1}{5}$

Answer: (A).

13. If $a = 2012$, $b = 2011$, $c = 2010$ then the value of $a^2 + b^2 + c^2 - ab - bc - ca$ is

(A) 0 (B) 2012 (C) 3 (D) 4024

Solution: $a = 2012 \quad b = 2011 \quad c = 2010$.

Let $a = b + 1 \qquad c = b - 1$.

Given expression is $= (b + 1)^2 + b^2 + (b - 1)^2 - b(b + 1) - b(b - 1) - (b - 1)(b + 1) = 3b^2 + 2 - 3b^2 + 1 = 3$.

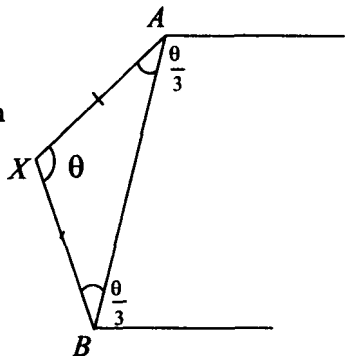
Answer: (C).

14. AX and BX are two adjacent sides of a regular polygon. If $\angle ABX = \frac{1}{3}\angle AXB$, then the number of sides of the polygon is

(A) 6 (B) 7 (C) 9 (D) 5

Solution: From the figure we have

$$\frac{5\theta}{3} = 180^\circ \Rightarrow \theta = \left(\frac{180 \times 3}{5}\right)^\circ$$



In a regular polygon of n sides each interior angle $= \dfrac{(n-2)180}{n}$

$\therefore \quad \dfrac{(n-2)180}{n} = \dfrac{180 \times 3}{5}$

$\Rightarrow 5n - 10 = 3n \Rightarrow n = 5$,

Answer: (D).

15. A rectangular block of dimensions $16 \times 10 \times 8$ units is painted. It is cut in to cubes of dimensions $1 \times 1 \times 1$. The number of cubes which are not painted at all is

(A) 945 (B) 672 (C) 812 (D) 796

Solution: Number of cubes not painted

$$= 14 \times 8 \times 6 = 672.$$

Answer: (B).

PART - B

1. The value of $\sqrt[3]{5+2\sqrt{13}} + \sqrt[3]{5-2\sqrt{13}}$ is = ———

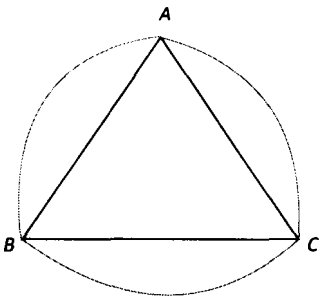
Solution: Let $a = \sqrt[3]{5+2\sqrt{3}} \quad b = \sqrt[3]{5-2\sqrt{3}}$.

We have the identity

$$\begin{aligned}
& (a+b)^3 = a^3 + b^3 + 3ab(a+b) \\
\Rightarrow \quad & (a+b)^3 = 5 + 2\sqrt{13} + 5 - 2\sqrt{13} \\
& \quad +3 \cdot \sqrt[3]{5^2 - (2\sqrt{13})^2} \cdot (a+b) \\
& \quad 10 + 3 \cdot \sqrt[3]{-27}(a+b) \\
\Rightarrow \quad & x^3 = 10 - 9x \quad \text{where } x = a+b) \\
\Rightarrow \quad & x^3 + 9x - 10 = 0 \\
\Rightarrow \quad & x^3 - 1 + 9x - 9 = 0 \\
\Rightarrow \quad & (x-1)(x^2 + x + 1) + 9(x-1) = 0 \\
\Rightarrow \quad & (x-1)(x^2 + x + 10) = 0 \\
\Rightarrow \quad & (x-1)[(x + \frac{1}{2})^2 + 9\frac{3}{4}] = 0 \\
\Rightarrow \quad & x = 1 \quad \text{is the only real solution.}
\end{aligned}$$

$\therefore \quad a+b=1$ Answer: $a+b=1$.

2. Triangle ABC is equilateral of side length 8cm. Each arc shown in the diagram is an arc of a circle with the opposite vertex of the triangle as its centre. The total area enclosed within the entire figure shown (in cm^2) ———



Solution: Area enclosed by the figure (in cm^2)

$=$ Area of sectors with centre A, B, C and radius BC or CA or AB $-$ 2 area of $\triangle ABC$

$$= 3 \times \frac{60}{360} \times \pi \times 8^2 - 2\left(\frac{\sqrt{3}}{4} \times 8^2\right)$$

$$= \frac{64\pi}{2} - \frac{64\sqrt{3}}{2} = 32(\pi - \sqrt{3})$$

Answer: $32(\pi - \sqrt{3})$.

3. If a, b, c, d, satisfy the equations $a + 7b + 3c + 5d = 0$, $8a+4b+6c+2d = -16$, $2a+6b+4c+8d = 16$, $5a+3b+7c+d = -16$ then the value of $(a + d)(b + c) =$ ———

Solution: Given that

$$a + 7b + 3c + 5d = 0 \tag{1}$$

$$8a + 4b + 6c + 2d = -16 \tag{2}$$

$$2a + 6b + 4c + 8d = 16 \tag{3}$$

$$5a + 3b + 7c + d = -16 \tag{4}$$

$(2) + (3) - (1) - (4) \Rightarrow 4a + 4d = -16 + 16 - 0 + 16 = 16$

$$a + d = 4 \tag{5}$$

$(1) + (2) + (3) + (4) \Rightarrow 16(a + d) + 20(b + c) = -16$

$\Rightarrow 64 + 20(b + c) = -16$

$$\Rightarrow\ 20(b+c) = -80$$
$$\Rightarrow\ b+c = -4.$$
$$\therefore\ (a+d)(b+c) = 4 \times (-4) = -16.$$

4. Lines $L_1, L_2, L_3, \cdots, L_{20}$ are distinct. All the lines L_4, L_8, L_{12}, L_{16} and L_{20} are parallel. All the lines $L_1, L_5, L_9, L_{13}, L_{17}$ pass through a given point A. the maximum number of points of intersection of these 20 lines is ———

Solution: Two lines (or) a pair of straight lines can intersect at a maximum of 1 point. One can select $\frac{20\times19}{2}$ pair of straight lines from the 20 lines $L_1, L_2, L_3, \cdots, L_{19}, L_{20}$. Each of these 190 pairs of straight lines can obtain one intersection point giving rise to maximum of 190 intersection points. But 5 of these 20 lines L_4, L_8, L_{12}, L_{16} and L_{20} are parallel.

There will be no intersection point from any pair of straight lines from these 5 lines. $\therefore$ We need to remove $\frac{5\times4}{2}$ intersection points from 190. All the lines $L_1, L_5, L_9, L_{13}, L_{17}$ pass through only one point.

$\therefore$ We need to replace $\frac{5\times4}{2}$ intersection points by 1.

The maximum number of intersection points of these 20 lines is $190 - 10 - 10 + 1 = 171$.

Answer: 171.

5. x, y, z are real numbers such that $(x+y)^2 = 16, (y+z)^2 = 36$, $(z+x)^2 = 81$, $x+y+z > 3$. The number of possible values of $(x+y+z)$ is ————— .

Solution: $x+y = \pm4$, $y+z = \pm6$, $z+x = \pm9$ various combinations of $2(x+y+z)$ are $19, 11, 7, 1, -1, 7, -11, -19$.
$\therefore\ x+y+z = \frac{7}{2}, \frac{11}{2}$ or $\frac{19}{2}$. $\because\ x+y+z > 3$.
Answer: 3.

6. $ABCD$ is a square. A line AX meets the diagonal BD at X and $AX = 2012$ cm. the length of CX (in cm) is ———

Solution: In the triangles ABX, CBX

$AB = BC$

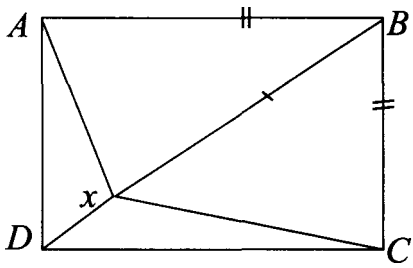
$\angle ABX = \angle CBX = 45°$

BX is common.

$\Rightarrow \triangle ABX \equiv \triangle CBX$

$\Rightarrow CX = AX = 2012.$

Answer: 2012.



7. A two digit number is 6 times the sum of its digits. The number formed by interchanging the digits is k times the sum of the digits. Then value of k is ———

Solution: $10a + b = 6(a + b) \Rightarrow 4a = 5b.$

The number formed by interchanging digits is $10b + a$

$$\begin{aligned} 10b + a &= k(a + b) \\ 8a + a &= k\left(a + \frac{4a}{5}\right) \\ 9a &= k\frac{9a}{5} \Rightarrow k = 5. \end{aligned}$$

Answer: 5.

8. O is the centre of a circle of radius 15cm. M is a point at a distance of 5cm from O. AMB is any chord of the circle through M, then the value of $AM \times MB$ is ———

Solution: Let $CMOD$ be the diameter

$AM \cdot MB = CM \cdot MD$

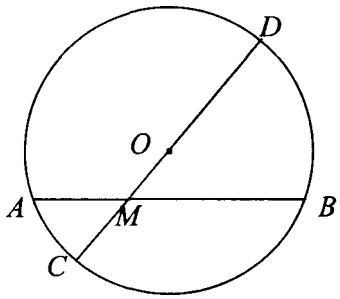
$= (CO - OM)(CO + OM)$

$= CO^2 - OM^2$

$= 225 - 25$

$= 200\text{cm}$

Answer: 200.



9. The combined age of a man and his wife is six times the combined ages of their children. Two years ago their united ages were ten times the combined ages of their children. Six years hence their combined age will be there times the combined age of the children. The number of children they have is ———

Solution: Let x be the combined age of man and his wife. Let there be n children and y be their combined age.

As per problem, $x = 6y$ (1)

$$x - 4 = 10(y - 2n) \quad (2)$$

(This equation is valid only if Ten years ago is replaced by Two years ago)

$$x + 12 = 3(y + 6n) \quad (3)$$

Using (1) and (2) $\Rightarrow$ $6y - 4 = 10(y - 2n) \Rightarrow 3y - 2 = 5(y - 2n)$

$$\Rightarrow 2y = 10n - 2 \quad (4)$$

Using (1) and (3) $6y + 12 = 3(y + 6n) \Rightarrow 2y + 4 = y + 6n$

$$\Rightarrow y = 6n - 4 \quad (5)$$

Using (5), (4) $\Rightarrow 2(6n - 4) = 10n - 2$

$$6n - 4 = 5n - 1 \Rightarrow n = 3$$

Number of children =3.

Answer: 3.

10. A two digit number is less than the sum of the squares of its digits by 11 and exceeds twice the product of its digits by 5. The two digit number is ———

Solution:

Let the two digit number be ab, i.e., $10a + b$

Given that $a^2 + b^2 - (10a + b) = 11$ (1)

and $10a + b - 2ab = 5$ (2)

adding (1) and (2)
$$\begin{aligned} a^2 + b^2 - 2ab &= 16 \\ (a-b)^2 &= 16 \\ a - b &= \pm 4. \end{aligned}$$

If $a = b + 4$, then, Eqn. (2) becomes

$$\begin{aligned} 10(b+4) + b - 2b(b+4) &= 5 \\ 10b + 40 + b - 2b^2 - 8b - 5 &= 0 \\ \text{i.e., } -2b^2 + 3b + 35 &= 0 \\ \text{i.e., } 2b^2 - 3b - 35 &= 0 \\ 2b^2 - 10b + 7b - 35 &= 0 \\ 2b(b-5) + 7(b-5) &= 0 \\ b &= 5 \text{ or } -7/2. \end{aligned}$$

If $a = b - 4$, we get $b = 5$ or $9/2$ since b is an integer $b = 5$ and $a = 9$. $\therefore$ The number is 95.
Answer: 95.

11. An isosceles trapezoid is circumscribed about a circle of radius 2cm and the area of the trapezoid is 20cm^2. The equal sides of the trapezoid have length ———

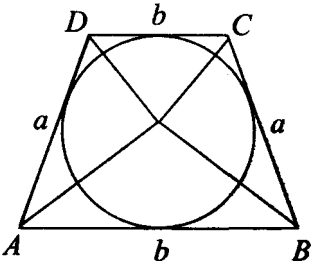
Solution: Area of trapezoid
$= \frac{1}{2}(b+c) \times 4 = 20$
$\Rightarrow \quad b + c = 10 \qquad (1)$

Area of trapezoid $= \frac{1}{2}c(2) + \frac{1}{2}a(2) + \frac{1}{2}b(2) + \frac{1}{2}a(2) = 20$

$\Rightarrow \quad b + c + 2a = 20 \quad (2)$

$\Rightarrow \quad 10 + 2a = 20 \quad$ (using (1))

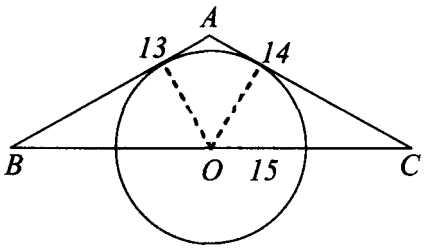
$a = 5$



Answer: 5cm

12. A triangle has sides with lengths 13cm, 14cm, and 15cm. A circle whose centre lies on the longest side touches the other two sides. The radius of the circle is (in cm) ———

Solution:



$$\begin{aligned} s &= \frac{13+14+15}{2} = 21 \\ \triangle &= \sqrt{s(s-a)(s-b)(s-c)} = 84 \\ &= \frac{1}{2} \times 13 \times r + \frac{1}{2} \times 14 \times r \\ \Rightarrow r &= 56/9\text{cm}. \end{aligned}$$

Answer: $56/9$.

13. The sum of the roots of the equation $x^{\sqrt[3]{x^2}} = (\sqrt{x})^x$ is ———

Solution:

$$x^{x^{2/3}} = x^{x/2} \tag{1}$$

$\Rightarrow x = 1$ or $x^{2/3} = x/2 \Rightarrow x^2 = \frac{x^3}{8}$

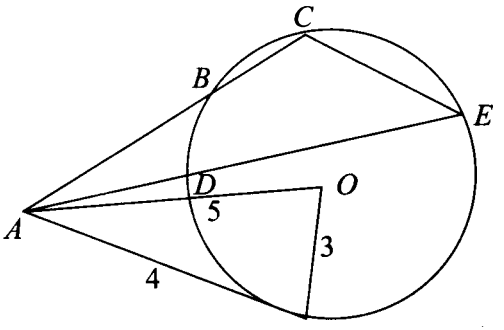
$\Rightarrow x^3 - 8x^2 = 0 \Rightarrow x^2(x-8) = 0 \Rightarrow x = 0, x = 8.$

The sum of the roots $= 8 + 1 = 9$.

Answer: 9.

14. ABC and ADE are two secants of a circle of radius 3cm. A is at a distance of 5cm from the centre of the circle. The secants include an angle of $30°$. The area of the $\triangle ACE$ is 10cm^2. Then the area of the $\triangle ADB$ (in cm^2) is ———

Solution:



Clearly the length of tangent from A to the circle is 4 cm.

By intersecting chord theorem,

$$AB \cdot AC = AD \cdot AE \qquad (1)$$

$$= 4^2 = 16$$

Area of $\triangle AEC = 10 = \frac{1}{2} AE \cdot AC \sin 30 = \frac{AE \cdot AC}{4}$

$\Rightarrow \; AE \cdot AC = 40$

$$\begin{aligned} \text{Now} \qquad & AB \cdot AC \cdot AD \cdot AE = 256 \\ & AB \cdot AD \cdot AC \cdot AE = 256 \\ & AB \cdot AD = 256/40 \end{aligned}$$

Area of $\triangle ABD = \frac{1}{2} \cdot AB \cdot AD \sin 30$

$$= \frac{AB \cdot AD}{4} = \frac{256}{4 \times 40} = \frac{16}{10} = \frac{8}{5}$$

Answer: $\frac{8}{5}$.

15. The value of x which satisfies the equation $5^2 \cdot 5^4 \cdot 5^6 \cdots 5^{2x} = (0.04)^{-28}$ is ———

Solution:

$$\begin{aligned} 5^2 \cdot 5^4 \cdot 5^6 \cdots 5^{2x} &= 5^{2+4+6+\cdots+2x} \\ &= 5^{2(1+2+3+\cdots+x)} = 5^{2 \cdot \frac{x(x+1)}{2}} \\ \Rightarrow \quad & 5^{x^2+x} = (0.04)^{-28} \\ \Rightarrow \quad & 5^{x^2+x} = (\frac{1}{25})^{-28} = 5^{56} \\ \Rightarrow \quad & x^2 + x - 56 = 0 \\ & (x+8)(x-7) = 0 \\ & x = 7. \end{aligned}$$

Answer: 7.

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; the never-satisfied man is so strange if he has completed a structure, then it is not in order to dwell in it peacefully, but in order to begin another. I imagine the world conqueror must feel thus, who, after one kingdom is scarcely conquered, stretches out his arms for others.

Karl Friedrich Gauss

SCREENING TEST- RAMANUJAN CONTEST
(NMTC at INTER LEVEL XI & XII Standards)

PART - A

1. The sum of the squares of all real numbers satisfying the equation $x^{256} - 256^{32} = 0$ is

   (A) 8 (B) 128 (C) 512 (D) 65536

   Solution:

   $x^{256} - 256^{32} = 0$

   $x^{256} - (2^8)^{32}$

   $x^{256} - 2^{256} = 0$

   $\Rightarrow \quad x = \pm 2$ are the two real solutions

   $\therefore \quad (2)^2 + (-2)^2 = 8.$

   Answer: (A)

2. Two adjacent vertices of a square are on a circle of radius R and the other two vertices lie on a tangent to the circle. The length of the side of the square is

   (A) $\frac{4R}{3}$ (B) $\frac{6R}{5}$ (C) $\frac{8R}{5}$ (D) $\frac{3R}{2}$

   Solution: From the diagram

   we have, $R^2 - (R - x)^2 = \frac{a^2}{4}$

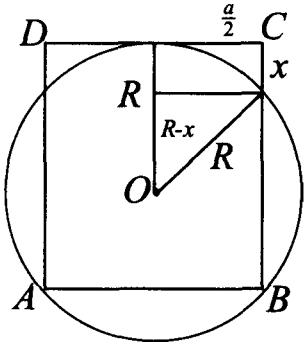
   $\Rightarrow \quad 2Rx - x^2 = \frac{a^2}{4}$

   $\Rightarrow \quad 4x^2 - 8Rx + a^2 = 0$ (1)

   By secant-tangent theorem,

   we have, $x \cdot a = \frac{a^2}{4}$

   $\Rightarrow \quad x = \frac{a}{4}$



   Eqn (1) becomes, $4\frac{a^2}{16} - 8R\frac{a}{4} + \frac{a^2}{16} = 0 \quad \Rightarrow \quad a = \frac{8R}{5}$.

   Answer: (C)

3. The polynomial $x^{2n}+1+(x+1)^{2n}$ is not divisible by (x^2+x+1) if n is equal to

(A) 17 (B) 20 (C) 21 (D) 64

Solution:

Let $x^{2n}+1+(x+1)^{2n} = Q(x)(x^2+x+1)$

$Q(x)$ is a polynomial with integer coefficients.

Let $x = 2$, we get, $2^{2n}+1+3^{2n} = Q(2)\cdot 7$ (1)

$\Rightarrow$ $7/2^{2n}+3^{2n}+1$.

If n is a multiple of 3, (i.e.), Let $n = 3k$ for some integer k,

$2^{2n} = 2^{6k} = (2^6)^K = 64^K = (63+1)^K \Rightarrow$ multiple of $63+1$

$3^{2n} = 3^{6k} = (3^6)^K = 729^K = (728+1)^K \Rightarrow$ multiple of $728+1$.

But 63 and 728 are multiple of 7. $\therefore$ $2^{2n}+3^{2n}+1 =$ a multiple of $7+3$ and will not satisfy (1).

$\therefore$ $x^{2n}+1+(x+1)^{2n}$ is not divisible by x^2+x+1 when n is a multiple of 3.

$\therefore$ Correct option is (C).

Answer: (C)

4. The remainder when 3^{302} is divided by 11 is

(A) 10 (B) 9 (C) 8 (D) 7

Solution: By Fermat's little theorem

$$\begin{aligned} 3^{10} &\equiv 1(\text{mod}11) \\ \Rightarrow\ 3^{300} &\equiv 1(\text{mod}11) \\ \Rightarrow\ 3^{302} &\equiv 9(\text{mod}11) \end{aligned}$$

$\therefore$ when 3^{302} is divided by 11 the remainder is 9.

Answer: (B)

5. If a, b and d are the lengths of a side, a shortest diagonal and a longest diagonal of a regular nonagon, then

   (A) $d^2 = a^2 + ab + b^2$ (B) $d^2 = a^2 + b^2$
   (C) $d = a + b$ (D) $b^2 = ad$

   Solution: The chords of lengths a, b, d subtend central angles $40°, 80°$ and $160°$ respectively.

   $a = 2r \sin 20°$ $b = 2r \sin 40°$ $d = 2r \sin 80°$

   where r is the radius of the circle circumscribing the nanogram.

   $\sin 40 + \sin 20 = 2 \sin 30 \cos 10° = \cos 10° = \sin 80°$

   $\Rightarrow \quad 2r \sin 40 + 2r \sin 20 = 2r \sin 80$

   $\Rightarrow \quad a + b = d.$

   Answer: (C).

6. If a positive integer, after adding 100, becomes a perfect square, and also after adding 168, becomes another perfect square. (168 is added to the original positive integer), then the sum of the digits of the integer is

   (A) 12 (B) 11 (C) 8 (D) 7

   Solution: Let a be the positive integer from given conditions we have, $a + 100 = m^2$, $a + 168 = n^2$

   $n^2 - m^2 = 68 \Rightarrow (n + m) - (n - m) = 68 = 34 \times 2.$

   The only possibility of $(n - m)$ and $(m + n)$ is

   $$n - m = 2; \qquad n + m = 34$$

   $\therefore \quad n = 18, \quad a + 168 = n^2 \Rightarrow a = 156.$

   Sum of the digits is 12.

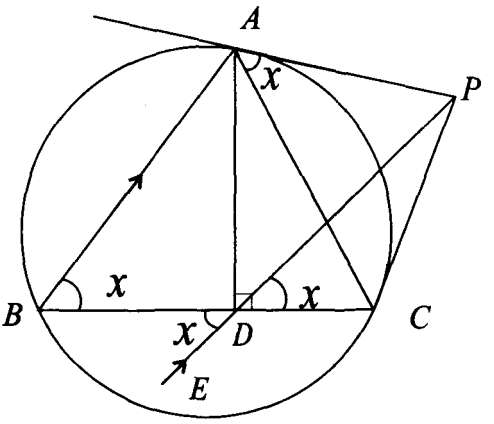
   Answer: (A)

7. ABC is a triangle inscribed in a circle. AD is the altitude of the triangle. DP is drawn parallel to AB to cut the tangent

at A at P. Then $\angle CPA$ is

(A) Equal to $90°$ when triangle ABC is acute angled
(B) Equal to $90°$ for any non-right angled triangle ABC
(C) Greater than $90°$ for the angle A obtuse
(D) Smaller than $90°$ for an acute angled triangle ABC.

Solution:



Let $\angle CDP = x$

$\angle BDE = x = \angle ABD$ $(\because AB||EDP)$

$\angle PAC = x$ (using alternates Segment theorem)

$ADCP$ is a cyclic quadrilateral

$\angle CPA + \angle ADC = 180°$

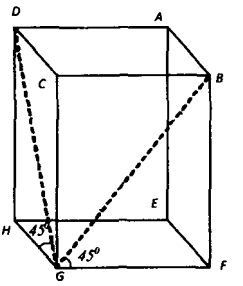
$\angle CPA + 90° = 180°$

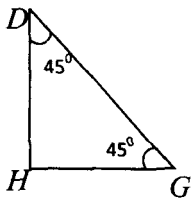
$\angle CPA = 90°$

Answer: (B)

8. In the adjoining figure of a rectangular solid, it is given that $\angle DGH = 45°$ and $\angle BGF = 60°$. Then $\cos(\angle BGD) =$

(A) $\frac{\sqrt{6}-\sqrt{2}}{4}$ (B) $\frac{\sqrt{2}}{6}$ (C) $\frac{\sqrt{6}}{4}$ (D) $\frac{\sqrt{3}}{6}$





Solution: $\angle DGH = 45°$ $\angle BGF = 60°$.

Let $GF = EH = BC = AD = 1$ unit

$BF = CG = DH = AE = \sqrt{3}$ units

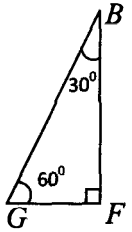
$HG = EF = AB = CD = \sqrt{3}$ units

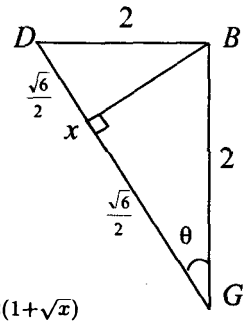
Draw altitude BX

$DX = XG = \frac{\sqrt{6}}{2}$

$\cos \angle BGD = \frac{XG}{BG} = \frac{\sqrt{6}}{4}$.

Answer: $\frac{\sqrt{6}}{4}$ (C).





9. For the inequation $(1.25)^{1-x} < (0.64)^{2(1+\sqrt{x})}$

(A) There is no real value of x exists
(B) There are infinitely many negative values of x exist
(C) Any real value of x greater than 25 satisfies the inequation
(D) There is exactly one positive and one negative integer only satisfying the equation

Solution:

$$\left(\frac{5}{4}\right)^{1-x} < \left(\frac{16}{25}\right)^{2(1+\sqrt{x})}$$

$$\left(\frac{4}{5}\right)^{x-1} < \left(\frac{4}{5}\right)^{4(1+\sqrt{x})}$$

Now $0 < \frac{4}{5} < 1 \Rightarrow x - 1 > 4(1 + \sqrt{x})$

$x - 4\sqrt{x} - 5 > 0 \Rightarrow (\sqrt{x} - \sqrt{5})(\sqrt{x} + 1) > 0$

$\Rightarrow \sqrt{x} > 5 \Rightarrow x > 25.$

Answer: (C)

10. The number $(5^6 - 10^4)$ is

(A) Negative (B) divisible by 10
(C) divisible by 100 (D) divisible by 9

Solution:

$$\begin{aligned} 5^6 - 10^4 &= (5^3)^2 - (10^2)^2 \\ &= (5^3 - 10^2)(5^3 + 10^2) \\ &= 25 \times 225 \\ &= 25^2 \times 9. \end{aligned}$$

It is divisible by 9.

Answer: (D)

11. There are 'a' points on a line and 'b' points on a parallel line. The number of triangles formed by these $(a + b)$ points as vertices is

(A) $\frac{ab(a+b-2)}{2}$ (B) $\frac{ab(a+b-1)}{2}$ (C) $\frac{ab(a+b-4)}{2}$ (D) $\frac{ab(a+b)}{2}$

Solution: Any two points of one line and any one point of the other line determines a triangle.

$$\begin{aligned} \text{Total number of triangles} &= a_{c_2} \cdot b_{c_1} + a_{c_1} \cdot b_{c_2} \\ &= \frac{ab}{2}(a + b + 2). \end{aligned}$$

Answer: (A)

12. The equation $x^4 + 16x - 12 = 0$ has

(A) all the roots real
(B) all the roots complex
(C) two roots real and two roots complex
(D) all the roots integers.

Solution: $x^4 + 16x - 12 = 0$

$(x^2 + 2)^2 - 4(x - 2)^2 = 0$

$\Rightarrow \quad (x^2 + 2x - 2)(x^2 - 2x + 6) = 0\,.$

The roots are $-1 \pm \sqrt{3},\ 1 \pm i\sqrt{3}\,.$

Answer: (C)

13. A positive integer n has 60 divisors and $7n$ has 80 divisors. What is the greatest value of k such that 7^k divides n.

(A) 1 (B) 2 (C) 3 (D) 0

Solution: Let $n = 7^K \cdot Q$ where Q is the product of primes, none of which is 7.
Let d be the number of divisors of Q. Then n has $(k+1)d$ divisors. $7n = 7^{K+1} \cdot Q$ $\therefore$ $7n$ has $(K+2)d$ divisors.

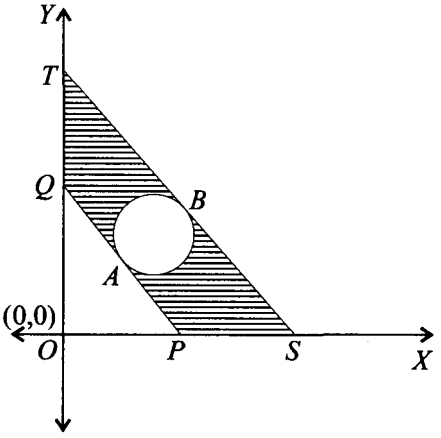
$$\text{Thus} \quad \frac{(K+2)d}{(K+1)d} = \frac{80}{60} = \frac{4}{3} \Rightarrow K = 2.$$

Answer: (B)

14. In the XY plane two points $A(2,2)$ and $B(7,7)$ are taken. R is the region in the first quadrant which consists of points C such that triangle ABC is an acute angled triangle. The closest integer to the area of the region R is

(A) 25 (B) 39 (C) 51 (D) 60

Solution:
For $\triangle ABC$ acute all the angles of the triangle must be acute. For A to be acute, C must lie above the line passing through A and $\perp$ to AB. The segment of that line in the first quadrant lies between $P(4,0)$ and $Q(0,4)$. For the angle to be acute, the point C must lie below the line through B and $\perp$r to AB. i.e., the segment of that line in the first quadrant lies between $S(14,0), T(0,14)$.



For C to be acute, point C must lie outside the circle of diameter AB.

Let O be the origin.

The region is shaded whose area =

$$\begin{aligned} &= \triangle OST - \triangle OPQ - \text{circle area} \\ &= \frac{1}{2}14^2 - \frac{1}{2}4^2 - \pi(\frac{\sqrt{50}}{2})^2 \\ &= 90 - \frac{25}{2}\pi \approx 51 \end{aligned}$$

Answer: (C).

15. How many three digit numbers have distinct digits such that one digit is the average of the other two?

(A) 104 (B) 112 (C) 256 (D) 12

Solution: The set 3-digits, of such a number can be arranged to form an increasing A.P.

There are 8 possible sequences with a common difference 1. Since the first term can be any of the digits 0 through 7. There are 6 possible sequences with a common difference of 2. 4 with common difference of 3, and 2 with common difference of 4. Thus there are 20 possible APs. Each of the 4 sets contain O can be arranged to form $2(2!) = 4$ different numbers. 16 sets that do not contain O can be arranged to form $3! = 6$ different numbers.

Total $= 4 \times 4 + 16 \times 6 = 112$.

Answer: (B).

PART - B

1. Let $f(x) = x^2 + bx + c$ where b, c are integers. If $f(x)$ is a factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$, then the value of $f(1)$ is ——— .

Solution: $f(x) = x^2 + bx + c$.

It is a factor of $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$.

$\therefore$ It must be a factor of

$3(x^4 + 6x^2 + 25) - 3x^4 + 4x^2 + 28x + 5$

$= 14x^2 - 28x + 70 = 14(x^2 - 2x + 5)$

$f(x) = x^2 - 2x + 5$

$f(1) = x^2 - 2 + 5 = 4$.

Answer: $f(1) = 4$.

2. Mahadevan's age is 'a' years, which is also the sum of the ages of his three children. His age b years ago was twice the sum of their ages. Then $\left(\frac{a}{b}\right) =$ ——.

Solution: Mahadevan's age 'b' years ago is $= a - b$

Sum of the ages of three children at that time is $= a - 3b$.

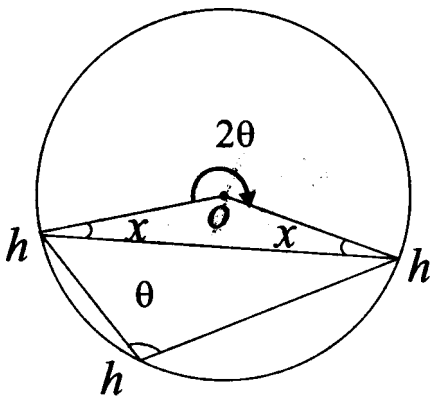
By data $a - b = 2(a - 3b)$

$\Rightarrow \frac{a}{b} = 5$.

Answer: 5.

3. $\angle RST$ is an angle in the minor segment of a circle of centre O. then the angle (in degrees) RST less the angle ORT is ——.

Solution:



$$\text{Let } \angle RST = \theta^\circ$$
$$\angle ORT = x^\circ$$

To find $(\theta - x)$.

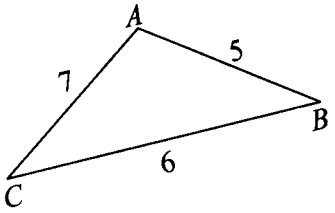
Clearly reflex $\angle ROT = 2\theta^\circ$

$\Rightarrow \quad \angle ROT = 360^\circ - 2\theta^\circ$

$$\text{In } \triangle ROT, 360 - 2\theta + 2x = 180^\circ$$
$$2(\theta - x) = 180^\circ$$
$$\theta - x = 90^\circ$$

Answer: 90°

4. A, B, C are three towns connected by straight roads from A to B, B to C and C to A. $AB = 5\,\text{km}$, $BC = 6\,\text{km}$ $CA = 7\,\text{km}$. Two cyclists start simultaneously form A and go in different roads with same speed. They meet at D. then $BD =$ ——.



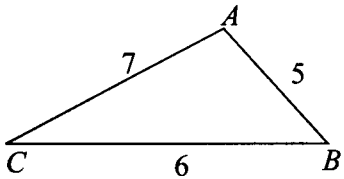
Solution: The perimeter of triangle is $= 5 + 6 + 7 = 18$.

The distance of each cyclist travel $= 9\,\text{km}$.

$\therefore \quad AB + BD = 9$.

$\therefore \quad BD = 4\,\text{km}$.

Answer: 4km.



5. f is a linear function given by $f(x) = ax + b$ and $f^{-1}(x) = bx + a$ when a, b are real. The value of $a + b$ is equal to ——.

Solution:

$$f(x) = ax + b = y(say)$$
$$\Rightarrow \quad x = \frac{y - b}{a}$$
$$\Rightarrow \quad f^{-1}(x) = \frac{x - b}{a} = \frac{x}{a} - \frac{b}{a} = bx + a$$

$$\Rightarrow \quad b = \frac{1}{a} \text{ and } -\frac{b}{a} = a$$
$$\Rightarrow \quad a^3 + 1 = 0 \Rightarrow (a+1)(a^2 - a + 1) = 0$$

a is real $\Rightarrow a = -1 \Rightarrow b = -1$.
$\therefore\ a + b = -1 - 1 = -2$.
Answer: -2.

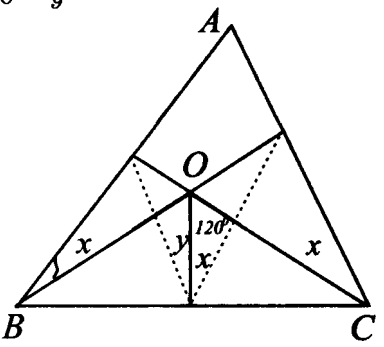
6 ABC is a triangle. A point O is taken inside the triangle such that $\angle BOC = 120°$. OD, OE, OF are drawn perpendiculars to the sides BC, CA, AB respectively. Then $\angle EDF + EAF =$ ———.

Solution: From the figure we have

$$\angle ODE = \angle OCE = x$$
$$\angle ODF = \angle OBF = y$$

($ODCE, ODBF$ are cyclic)

$$\angle EOC = 90 - x$$
$$\angle FOB = 90 - y$$



Angle at $O = 360 = 120 + 90 - x + 90 - y + \angle EOF$

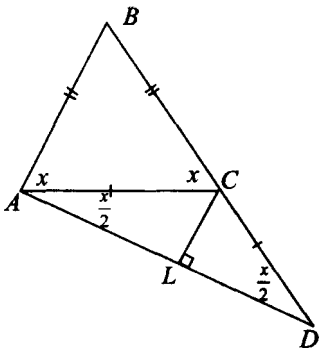
$\Rightarrow \quad \angle EOF + \angle A = 180°$ ($\because FOEA$ is cyclic)

$\Rightarrow \quad \angle EDF + \angle EAF = x + y + \angle EAF = 120°$

Answer: 120°

7. ABC is an isosceles triangle in which $AB = BC$. BC is produced to D such that $\angle CAD = \frac{1}{2}\angle BAC$. If L is the foot of the $\perp$r from C to AD, then the value of $\frac{AL}{AD} =$ ——.

Solution:



Let $\angle BAC = \angle BCA = x$

$\Rightarrow \quad \angle B = 180 - 2x$.

In $\triangle ABD$,

$\angle BDA + \frac{x}{2} + x + 180 - 2x = 180$

$\Rightarrow \quad \angle BDA = x/2$

$\Rightarrow \quad \triangle ACD$ is an isosceles $\triangle$.

The perpendicular CL to the base bisects the base

$\therefore \frac{AL}{AD} = \frac{AL}{2AL} = \frac{1}{2}$.

Answer: $\frac{1}{2}$.

8. The values of x satisfying the inequality $x - \sqrt{1 - |x|} < 0$ lie in $\left[-1, \frac{a}{2}\right]$. Then the value of a is ——.

Solution:

$$x - \sqrt{1 - |x|} < 0$$
$$\Rightarrow \quad x < \sqrt{1 - |x|}$$
$$\Rightarrow \quad x^2 < 1 - |x|$$
$$x^2 + |x| - 1 < 0$$

Case (i) $\quad x \geq 0$

$$x^2 + x - 1 < 0$$
$$(x + \tfrac{1}{2})^2 - \tfrac{5}{4} < 0$$
$$x < \tfrac{\sqrt{5}-1}{2}$$

Case (ii) $x < 0$

$x^2 - x - 1 < 0$

$(x - \frac{1}{2})^2 - \frac{5}{4} < 0$

$x < \frac{1+\sqrt{5}}{2}$

But $x = \frac{\sqrt{5}+1}{2}$ does not satisfy the equation.

Answer: $\sqrt{5} - 1$.

9. For some real numbers a and b, the equation $8x^3 + 4ax^2 + 2bx + a = 0$ has three distinct positive roots. If the sum of the logarithms to base 2 of the roots is 5, the value of a is ——.

Solution: The equation is $8x^3 + 4ax^2 + 2bx + a = 0$. Let α, β, λ be the roots.

Then, $\log_2 \alpha + \log_2 \beta + \log_2 \lambda = 5 = \log_2 \alpha\beta\lambda$

$\Rightarrow \quad \alpha\beta\lambda = 2^5 = 32 = \frac{-a}{8}$

$\Rightarrow \quad a = -256$.

Answer: -256.

10. $ABCD$ is a quadrilateral in the first quadrant of the coordinate axes. A is (3,9), B is (1,1) C is (5,3) and D is (a, b). The quadrilateral formed by joining the mid-points of AB, BC, CD and DA is a square. The sum of the coordinates of the point D is ——.

Solution: Let M, N, P, Q be the mid points of AB, BC, CD and DA respectively. M is (2,5), N is (3,2), slope of $MN = -3$.

$\therefore$ Slope of $MQ = \frac{1}{3}$

$MQ = MN = \sqrt{10}$, equation of MQ is $y - 5 = \frac{x-2}{3}$

$\therefore$ Q has coordinates $(a, \frac{a+13}{3})$ $MQ = \sqrt{10}$

$\Rightarrow \quad (a-2)^2 + (\frac{a+13}{3} - 5)^2 = 10$

$\Rightarrow$ On simplification $a - 2 = \pm 3$.

Q is in first quadrant $a = 5$.

$\therefore$ Q is (5,6) since Q is the mid point of AD and A is (3,9).

$\therefore$ D is (7,3). Sum of the coordinates of D is $7 + 3 = 10$.

Answer: 10.

11. The number of distinct four-tuples of numbers (a, b, c, d) of rational numbers satisfying $a\log_{10} 2 + b\log_{10} 3 + c\log_{10} 5 + d\log_{10} 7 = 2012$ is ——.

Solution: Given equation is equivalent to

$\log_{10}(2^a 3^b 5^c 7^d) = 2012$

$\Rightarrow \quad 2^a 3^b 5^c 7^d = 10^{2012} = 2^{2012} \cdot 5^{2012}$.

Let the common denominator of a, b, c, d has the least value m.

$2^{ma} 3^{mb} 5^{mc} 7^{md} = 2^{2012m} \cdot 5^{2012m}$

$$\begin{aligned} ma &= 2012m \\ mb &= 0 \\ mc &= 2012m \\ md &= 0. \end{aligned}$$

$\therefore$ The only one solution is $(a, b, c, d) \equiv (2012, 0, 2012, 0)$.

Answer: 1.

12. If the graph of $f(x) = \big||x-2| - a\big| - 3$ has exactly three x - intercepts, then a is equal to ——.

Solution: $\big||x-2| - a\big| = 3$.

$$\Rightarrow \quad \begin{array}{cc} |x-2| - a = 3 \text{ or } & |x-2| - a = -3 \\ |x-2| = 3 + a \text{ or } & |x-2| = a - 3 \\ \downarrow & \downarrow \\ (1) & (2) \end{array}$$

If $a = 3$ then (2) $\Rightarrow$ $\boxed{\text{x=2}}$

If $a < 3$ then (2) has no solution.

To have 3 solutions (1) should have two solutions. For $a = 3$ (1) $\Rightarrow$ $|x - 2| = 6$ it has two solutions.

But if $a < 3$ (1) has 2 solutions. We want 3 solutions. Thus a can not be less than 3. Let us take the case $a > 3$.

Then (1) and (2) will have each two solutions. Thus there will be 4 solutions.

$\therefore$ To have only 3 solutions $a = 3$.

13. If we add the square of the digit in the tens place of a positive two digit number to the product of the digits of the number, we get 52. If we add the square of the digit in the units place of the number to the same product of the digits, we get 117. The two digit number is ——.

Solution: Let $10a + b$ be the two digit number.

Given:

$$\left.\begin{aligned} ab + a^2 &= 52 \\ ab + b^2 &= 117 \end{aligned}\right\} \Rightarrow \begin{aligned} (a+b)^2 &= 169 \Rightarrow a + b = 13 \\ a(a+b) &= 52 \Rightarrow a = 4 \end{aligned}$$

$\therefore$ $b = 9$.

The two digit number is 49.

I am interested in mathematics only as a creative art.

G.H. Hardy

FINAL - GAUSS CONTEST
(NMTC at PRIMARY LEVEL V to VI Standards)

1. Find the sum (S) of all numbers with 2012 digits and digital sum 2. Find also the digital sum of S.

Solution:

(a) Consider numbers consisting of the leading digit '2' and all the other 2011 digits on the right of 2, zero. The only one such number is $2\underbrace{0\,0\,0\cdots0\,0}_{2011\text{ zeros}}$

(b) The leading digit '1', and of the other 2011 digits, one digit is one and the rest of 2010 digits, zero. The second one besides the leading digit may occupy any one of the 2011 digits. Therefore there are 2011 chance for the second one. Thus there are 2011 numbers each of which having two ones and 2010 zeros. These 2011 numbers are

1. $1\underbrace{0\,0\,0\cdots0}_{2010\text{ zeros}}1$
2. $1\underbrace{0\,0\cdots0}_{2009\text{ zeros}}1\,0$
3. $1\underbrace{0\,0\cdots0}_{2008\text{ zeros}}1\,0\,0$

$\cdots$ .....

$\cdots$ .....

2011. $1\,1\underbrace{0\,0\cdots0}_{2010\text{ zeros}}$

Hence there are 2012 numbers on total.

The sum of all 2012 numbers= $2013\underbrace{1\,1\,1\cdots1\,1}_{2011\text{ ones}}$

$\therefore$ Digital sum of the answer= $2+0+1+3+2011=2017$.

2. A number 'n' is called a "lonely odd composite" number if

   (a) n is an odd composite number and
   (b) both $(n-2)$ and $(n+2)$ are prime numbers.

   {n is odd, hence $n-2, n, n+2$ are three consecutive odd numbers.
   Ex. $n = 105$. Then $n-2 = 103$ and $n+2 = 107$. Here 103 is an odd composite number 103 and 107 are prime numbers. Therefore 105 is a "lonely odd composite" number }.

   Find all "lonely odd composite" numbers less than 100. Show that these lonely odd composite numbers are multiples of 3.

   Solution: The first odd composite number that is also a lonely number is 9, since $9-2, \boxed{9}, 9+2$ gives $7, \boxed{9}, 11$, where 7 and 11 are prime. 15 is the next lonely number as $13, \boxed{15}, 17$ forms the triples with 13 and 17 are prime numbers and . $13, \boxed{15}, 17$ are three consecutive odd numbers. The next three consecutive odd numbers, the middle of which is lonely is

   (3) $19, \boxed{21}, 23$.

   25,27,33 and 35 are odd composite numbers, but they are not lonely (why?).

   The list here gives the lonely numbers greater than 21 and less than 100

   (4) $37, \boxed{39}, 41$ (5) $43, \boxed{45}, 47$ (6) $67, \boxed{69}, 71$
   (7) $79, \boxed{81}, 83$ (8) $97, \boxed{99}, 101$.

   Thus there are exactly 8 lonely numbers less than 100.

   All these 8 eight lonely numbers listed above are multiples of 3.

   From the definition of lonely numbers $n, w_i(n(n-2))$ and $(n+2)$ to be prime. The preceding number $(n-2)$ as well as the succeeding number $(n+2)$ are prime numbers and hence not divisibly by any other number other than 1 and themselves. So, $(n-2)$ is not divisible by 3 ($n-2$ is greater than 3 as

the first lonely number itself is 9). Let this number be $3m + 1$ (i.e., $(n - 2)$ leaves a remainder 1 on dividing by 3).

Then $n = (n - 2) + 2 = 3m + 1 + 2 = 3m + 3 = 3(m + 1)$ so, the lonely number is divisible by 3 and $n + 2 = n - 2 + 4 = 3m + 1 + 4 = 3m + 3 + 2 = 3(m + 1) + 2$ and $(n + 2)$ leaves a remainder 2 on dividing by 3.

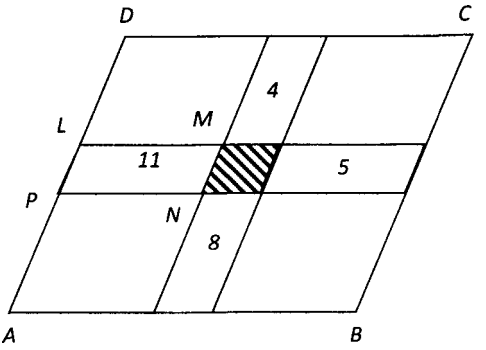
If $(n - 2)$ leaves a remainder 2 then $n - 2$ can be written as $3m + 2$ and $n = 3m + 4$ is not divisible by 3 (why?) and $n + 2 = 3m + 4 + 2 = 3m + 6 = 3(m + 2)$ and it divisible by 3.

But $(n+2)$ should be a prime number and hence not divisible by 3. We have assumed that $n - 2, \boxed{\text{n}}$ and $n + 2$ are three consecutive odd numbers and both $(n - 2)$ and $(n + 2)$ are prime numbers and this implies that $(n-2)$, the prime number preceding the lonely number always leaves $(n + 2)$ succeeding the lonely number always leaves a remainder 2 on dividing by 3.

Note: Given: Three consecutive odd numbers, exactly one of them is always divisible by 3. If the left most number leaves a remainder 1 on dividing by 3, then the middle number is divisible by 3, and the right most number leaves a remainder 2 on dividing by 3.

3. In the adjoining figure $ABCD$ is a parallelogram of perimeter 21.

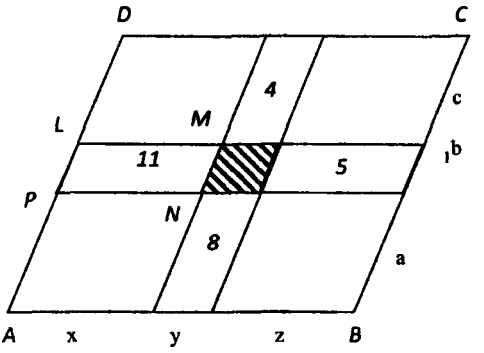
It is subdivided into smaller parallelograms by drawing lines parallel to the sides.



The numbers shown are the respective perimeters of the parallelograms in which they are marked. (For example the perimeter of the parallelogram $LMNP$ is 11). Find the perimeter of the shaded parallelogram.

Solution: Let x, y, z, a, b, c be the lengths of the sides of the respective parallelogram in the diagram. Clearly, the following equations hold

$$2(x+b)=11, \quad 2(z+b)=5, \quad 2(y+a)=8, \quad 2(y+c)=4.$$



We have to find $2(y+b)$.

Adding all the equations we get,

$$2(x+b+z+b+y+a+y+c)=28 \tag{1}$$

From given, perimeter of $ABCD=21$, we get

$$2(x+y+z+a+b+c)=21 \tag{2}$$

(1) - (2) $\Rightarrow \quad 2(y+b)=7.$

$\therefore$ Perimeter of the shaded portion=7.

4. l and b are two numbers of the form $\frac{p}{q}$ where p and q are natural numbers. Further l, b are greater than 2.

(a) If $l=\frac{2b}{b-2}$ prove that $b=\frac{2l}{l-2}$. (b) When l takes three values 3,4 and 8 find the corresponding values of b. (c) When b takes three values 6,10,12 find the corresponding values of l. **(d) From (b) and (c) we get 6 sets of values of (l, b).**

Taking l, b are the length and breadth of a rectangle, find the perimeter and area of the 6 rectangles. What is your inference?

Solution:

$$\begin{aligned} l &= \frac{2b}{(b-2)} \\ l \times (b-2) &= 2b \quad \text{(multiplying both sides of (1) by } (b-2)) \\ lb - 2l &= 2b \\ lb - 2b &= 2l \\ (l-2)b &= 2l \\ \therefore\ b &= \frac{2l}{l-2} \end{aligned}$$

Values of l	3	4	8
Values of b	6	4	$\frac{8}{3} = 2\frac{2}{3}$
Values of b	6	10	12
Values of l	3	$2\frac{1}{2}$	$2\frac{2}{5}$

In the above cases, we get 6 pairs of values for l, b and they are

$(3,6), (4,4), (8, 2\frac{2}{3})$

$(3,6), (2\frac{1}{2}, 10), (2\frac{2}{5}, 12)$

$l \times b$	=	18,	16,	$\frac{64}{3} = 21\frac{1}{3}$	18,	25,	$\frac{144}{5} = 28\frac{4}{5}$
$2(l+b)$	=	18,	16,	$\frac{64}{3} = 21\frac{1}{3}$	18,	25,	$\frac{144}{5} = 28\frac{4}{5}$

If the above number pairs are taken as length and breadth of rectangles these rectangle have the same perimeter and area numerically.

Thus there may be an infinite number of rectangles whose perimeter and areas are numerically equal.

5. **a, b, c, d are the units digits of four natural numbers each of which has four digits. The tens digit of these four numbers are the 9 complements of the units digit. The hundreds digits are the 18 complements of the sum of their respective tens and**

units digits. The thousands digits are the 27 complements of the sum of their respective hundreds, tens and units digits. If $a+b+c+d=10$, find the sum of these four numbers. {9 complement of a number $4x$ is $9-x$, 18 complement of a number y is $18-y$, 27 complement of a number z is $27-z$}.

Solution: The four numbers are:

		1000*s*	100*s*	10*s*	*units*
(1)		$27-18$	$18-9$	$9-a$	a
	$=$	9	9	$9-a$	a
(2)		9	9	$9-b$	b
(3)		9	9	$9-c$	c
(4)		9	9	$9-d$	d
Sum		36	36	$36-(a+b+c+d)$	$(a+b+c+d)$

$(a+b+c+d)=10$

gives

	36	36	26	10
Unsig the carry over, we get, 3	9	8	7	0

The answer is 39870.

6. A sequence is generated starting with the first term t_1 as a four digit natural number. The second third and fourth terns (t_2, t_3 and t_4) are got by squaring the sum of the digits of the preceding terms. (Ex. $t_1=9999$ then $t_2=(9+9+9+9)^2=36^2=1296$, $t_3(1+2+9+6)^2=324$, $t_4=(3+2+4)^2=81$). Start with $t_1=2012$. Form the sequence and find the sum of the first 2012 terms.

Solution:

$T_1=2012$ $\quad T_2=5^2=25$ $\quad T_3=7^2=49$

$T_4=13^2=169$ $\quad T_5=16^2=256$ $\quad T_6=13^2=169$

Thus the term of this sequence starts responding in pairs from t_4.

i.e., t_1, t_2 and t_3 are all distinct numbers and $t_4 = t_6 = t_8 = \cdots = t_{2012} = 169$ and $t_5 = t_7 = t_9 = \cdots = t_{2011} = 256$. There are 1005 terms each of which is 169 and there are 1004 terms each of which is 256.

So the sum of the sequence is

$$\begin{aligned} 2012 + 25 + 49 + 1004(169 + 256) + 169 \\ &= 2086 + 169 + 1004 \times 425 \\ &= 428955. \end{aligned}$$

7. Find the two digit numbers that are divisible by the sum of their digits. Give detailed solution with logical arguments.

Solution: Let ab be the two digit number. Clearly $a \neq 0$.

The number is $ab = 10a + b = 9a + (a + b)$ (1)

Sum of the digits $= a + b$ (2)

From the given condition, $(a + b)$ divides $10a + b$

$\Rightarrow$ $(a + b)$ divides $9a$.

If $b = 0$ then $a + b = a$ which divides $9a$.

$\therefore$ $10, 20, 30, 40, \cdots, 90$ are divisible by sum of their respective digits.

If $b \neq 0$ then $(a + b)$ should divide $9a$.

Case (1): If $a = 1$, then $1 + b$ should divide 9. The divisors of 9 greater than 1 are 3 and 9 and the corresponding values of b are 2 and 8.

$\therefore$ The numbers are 12 and 18.

Case (2): If $a = 2$, $9a = 18$ and $2 + b = 3, 6, 9$

$\Rightarrow$ The numbers are 21, 24 and 27.

Case (3): If $a = 3, 9a = 27$ and $a + b = 3 + b$. The divisors of 27 to be taken are 3,9.

$(3 + b = 3 \Rightarrow b = 0$ which is already counted) when it is 9, $b = 6$. $\therefore$ The number is 36.

Case (4): If $a = 4$ then $9a = 36$. The divisors of 36 to be considered are 6,9,12 and the corresponding values of b are $2, 5$ and 8. The numbers are 42, 45 and 48.

Case (5): If $a = 5$, $9a = 45$ and the suitable divisor is only 9 and the corresponding value of b is 4. The only number is 54.

Case (6): If $a = 6$, $9a = 54$ and the suitable divisor is 9 and the number is 63.

Case (7): If $a = 7$ and $9a = 63$, the only divisor to be considered is 9 and the corresponding number is 72.

Case (8): If $a = 8$ and $9a = 72$, the suitable divisors are 9 and 12 and the numbers are 81 and 84.

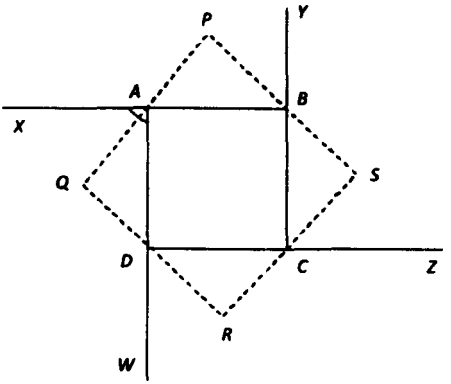
Case (9): If $a = 9$ and $9a = 81$, there is no divisors to be considered because they are already counted.

The numbers are 10, 20, 30, 40, 50, 60, 70, 80, 90

12, 18, 21, 24, 27, 36, 42, 45, 48, 54, 63, 72, 81, 84.

There are 23 numbers.

8. $ABCD$ is a square and the sides are extended as shown in the diagram. The exterior angles are bisected and the bisectors extended to from a quadrilateral $PQRS$. Prove that $PQRS$ is a square.



Solution:

(1) $\angle QAD + \angle DAB + \angle BAP = 45° + 90° + 45° = 180°$

[AQ bisect $\angle XAD$. $\therefore$ $\angle QAD = \angle QAX = \frac{90}{2} = 45°$]

(2) The points Q, A, P lie on the same line

(3) In $\triangle AQD; \angle QAD = \angle QDA = 45°$. (all the base angles of the four triangles formed on the sides of the square are equal by (1))

(4) $\therefore$ $AQ = QD$.

(Similarly $AP = PB$, $BS = SC$, $CR = RD$).

(5) Since all the four isosceles triangles are constructed on the sides of the square the length AQ, QD, AP, PB, BS, SC, CR and RD are equal.

(6) $\therefore$ $QP = QA + AP = PB + BS = PS = SC + CR = SR = RD + QD = RQ$.

The real danger is not that computers will begin to think like men, but that men will begin to think like computers.

Sydney J. Harris

FINAL - KAPREKAR CONTEST

(NMTC at SUB-JUNIOR LEVEL VII & VIII Standards)

1. Find the number of numbers coprime to and less than 2012. Find their sum. Find also the quotient when this sum is divided by 2012. (Information: 503 is a prime).

   Solution: $2012 = 4 \times 503$

   All even numbers are non coprime to 2012 besides 503 and 1509.

   The number of even numbers less than $2010 = \frac{2010}{2} = 1005$.

   The odd numbers having a common factor with 2012 are 503 and 1509.

   $\therefore$ The number of numbers which are non co-prime to 2012 and less than 2012 is $1005 + 2 = 1007$.

   The number of numbers less than 2012 is 2011.

   $\therefore$ The number of numbers co-prime to 2012 $= 2011 - 1007 = 1004$

   Sum of these 1004 co-prime number is sum of all these 1004 odd numbers or sum of all the odd numbers less than $2012 - (503 + 1509)$

   $2011 = (2 \times 1006 - 1)$.

   That is 2011 is 1006^{th} odd number and the sum to 1006 odd numbers is 1006^2.

   Sum of the co-prime numbers to 2012 is $1006^2 - 2012$

   $= 1006[1004] = 1005^2 - 1 = 1010025 - 1$.

   The quotient is 502.

2. Composite twins are defined below:

   (a) Odd composite twin: let a and $a+2$ be two odd composite numbers. If $(a-2)$ and $(a+4)$ are primes then $(a, a+2)$ is called an "odd composite twin".

(b) Even composite twin: Let b and $b+2$ two even numbers $(b > 2)$. If $(b-1)$ and $(b+3)$ are primes then $(b, b+2)$ is called an even composite twin.

List all composite twins less than or equal to 100.

Solution: Odd composite twins upto 100

(I) (1) (25,27) (2) (33,35) (3) (49,51) (4) (55,57)
(5) (63,65) (6) (75,77) (7) (85,87).

Even composite twins upto 100

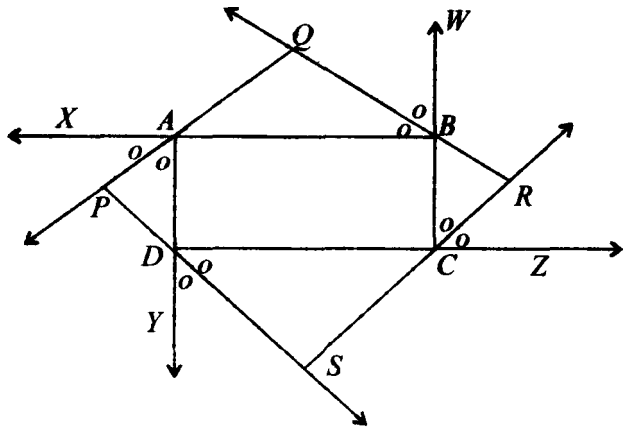
(1) (4,6) (2) (8,10) (3) (38,40) (4) (44,46)
(5) (68,70) (6) (98,100)

(II) In the case of odd composite twins the four numbers are of the form $(a-2, a, a+2, a+4)$. Their sum is $4a+4 = 4(a+1)$ and in the case of even composite twins the four numbers are of the form $(b-1), b, (b+2), (b+3)$. Their sum is $4b+4 = 4(b+1)$.

Thus the four number in either case is always a multiple of 4.

3. $ABCD$ is a rectangle. The sides are extended and the external angles are bisected and the bisectors are produced in both ways to form a quadrilateral. Prove that the quadrilateral is a square.

Solution:



$$\angle DAX = 90^\circ \quad [\angle DAB = 90^\circ, \angle DAB + \angle DAX = 180^\circ]$$

$$\angle DAP = \frac{1}{2}\angle DAX = 45^\circ$$

Similarly $\angle YDC = 90^\circ$, $\angle YDS = 45^\circ \angle PDA$

$\therefore$ In $\triangle APD, \angle PAD = \angle ADP = 45^\circ$

$$\therefore \angle APD = 180^\circ - (45^\circ + 45^\circ)$$
$$= 90^\circ$$

Similarly $\angle AQB = \angle BRC = \angle CSD = 90^\circ$

$\therefore$ $PQRS$ is a quadrilateral with all the angles equal, equal to 90°.

In $\triangle APD; \angle PAD = \angle PDA = 45^\circ$

$\therefore$ $\triangle$ is isosceles and right $\angle d$.

$\therefore$ $AP = PD = \frac{b}{\sqrt{2}}$

Similarly, $BR = CR = \frac{b}{\sqrt{2}}$ also $AQ = BQ = CS = DS = \frac{b}{\sqrt{2}}$

$\angle PAD + \angle DAB + \angle BAQ = 45^\circ + 90^\circ + 45^\circ = 180^\circ$

$\therefore$ P, A, Q is a straight line.

$\therefore$ $PQ = PA + PQ = \frac{b}{\sqrt{2}} + \frac{l}{\sqrt{2}} = (\frac{l+b}{\sqrt{2}})$

Similarly, $QR = RS = SP = (\frac{l+b}{\sqrt{2}})$.

$\therefore$ The quadrilateral $PQRS$ has all its angles and all its sides equal.

The area of $PQRS = (\frac{l+b}{\sqrt{2}})^2 = \frac{(l+b)^2}{2} = \frac{l^2+b^2+2lb}{2}$.

4. (a) A single digit natural number is increased by 10. The obtained number is now increased by the same percentage as in the first increase. The result is 72. Find the original single digit number.

(b) After two price reductions by one and the same percent the price of an article is reduced from Rs. 250 to Rs. 160. By how much percent was the price reduced each time. Write detailed steps.

Solution:

(a) Let the single digit number be x. After adding 10 it becomes $(10 + x)$.

The percentage of increase $= \frac{10}{x} \times 100 = \frac{1000}{x}\%$.

Given: The increased number is again increased by $\frac{1000}{x}\%$.

i.e., $(x + 10) + \frac{(x+10)1000}{x \times 100}$.

This number is given to be 72.

$$\therefore \quad (x + 10) + \frac{(x + 10)10}{x} = 72$$

$$\Rightarrow \quad (x + 10)\frac{(x + 10)}{x} = 72$$

$$(x + 10)^2 = 72x$$

$$\Rightarrow \quad x^2 - 52x + 100 = 0$$

$$\Rightarrow \quad (x - 2)(x - 50) = 0$$

$$\Rightarrow \quad x = 2 \text{ or } 50$$

The single digit number is 2.

(b) Let $x\%$ be the price reduction.

The first reduction $= 250 \times \frac{x}{100} = \frac{5x}{2}$ Rs.

The reduction price after the first reduction $= (250 - \frac{5x}{2})$ Rs.

The second reduction $= (250 - \frac{5x}{2})\frac{x}{100}$ Rs.

The price after the second reduction

$$= 250 - \frac{5x}{2} - (250 - \frac{5x}{2})\frac{x}{100} \text{ Rs.}$$

This is given to be 160 Rs.

$\therefore \quad (250 - \frac{5x}{2}) - (250 - \frac{5x}{2})\frac{x}{100} = 160$

$$\frac{(500-5x)}{2} - \frac{(500-5x)}{2}\frac{x}{100} = 160$$

$$(500 - 5x)[1 - \tfrac{x}{100}] = 2 \times 160.$$

$\Rightarrow$ $(500 - 5x)(100 - x) = 200 \times 160$

$\Rightarrow$ $5x^2 - 1000x + 18000 = 0$

$\Rightarrow$ $x^2 - 200x + 3600 = 0$

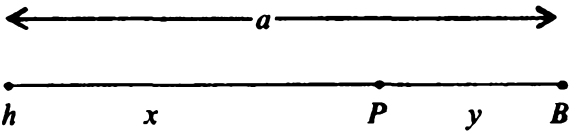
i.e., $(x - 20)(x - 180) = 0$

i.e., $x = 20$ or 180.

The percentage of reduction is 20 %

5. If a finite straight line segment is divided into two parts so that the rectangle contained by the whole and first part is equal to the square on the other part, prove that the square described on one of the diagonals of the rectangle contained by the whole and the first part is three times the square on the other part.

Solution:

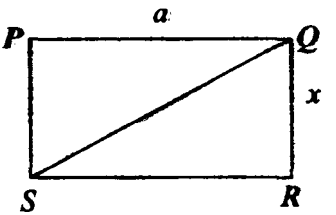


(a) Let $AB = a$ be the finite line segment. P divides it.

Let $AP = x, PB = y$.

Then $x + y = a$ (1)

Given: $ax = y^2$ (2)



Let $PQRS$ be the rectangle contained by the whole and the first part

$SQ^2 = a^2 + x^2$

To prove that $a^2 + x^2 = 3y^2$

(2) $\Rightarrow \quad ax = y^2$.

Using (1) we get $x(x+y) = y^2$

$\Rightarrow \quad x^2 + xy = y^2 \qquad (3)$

$$\begin{aligned} \text{Now, } a^2 + x^2 &= (x+y)^2 + x^2 \\ &= x^2 + 2xy + y^2 + x^2 \\ &= 2x^2 + 2xy + y^2 \\ &= 2(x^2 + xy) + y^2 \\ &= 2(y^2) + y^2 \qquad [\text{ Use (3)}] \\ &= 3y^2 \end{aligned}$$

6. *abcde* is a five digit number. Show that *abcde* is divisible by 7 if and only if the number $abcd - (2 \times e)$ is divisible by 7.

Solution:

$$\begin{aligned} \text{Let } x &= abcde \\ y &= abcd - 2 \times e \\ 10y + e &= 10abcd - 20e + e \\ &= abcd\ 0 + e - 20e \\ &= x - 20e. \end{aligned}$$

$$10y + 21e = x$$

$\Rightarrow \quad 7|x \Leftrightarrow 7|y$.

7. If $a^2x^3 + b^2y^3 + c^2z^3 = p^5, ax^2 = by^2 = cz^2$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{p}$ find $\sqrt{a} + \sqrt{b} + \sqrt{c}$ only in terms of p.

Solution: Let $ax^2 = by^2 = cz^2 = k$

$\Rightarrow \quad a = \frac{k}{x^2}, \ b = \frac{k}{y^2}, \ c = \frac{k}{z^2}$.

But $a^2x^3 + b^2y^3 + c^2z^3 = p^5$

$\Rightarrow \frac{k^2}{x^4} \cdot x^3 + \frac{k^2}{y^4} \cdot y^3 + \frac{k^2}{z^4} \cdot z^3 = p^5$

$\Rightarrow \frac{k^2}{x} + \frac{k^2}{y} + \frac{k^2}{z} = p^5$

i.e., $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{p^5}{k^2}$ (1)

But $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{p}$ (given)

$\Rightarrow \frac{1}{p} = \frac{p^5}{k^2} = k^2 = p^6$ or $k = p^3$

$$\begin{aligned}\sqrt{a} + \sqrt{b} + \sqrt{c} &= \frac{\sqrt{k}}{x} + \frac{\sqrt{k}}{y} + \frac{\sqrt{k}}{z} \\ &= \sqrt{k}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \\ &= \sqrt{k}\left[\frac{1}{p}\right] = \frac{p^{3/2}}{p} = \sqrt{p}.\end{aligned}$$

$\therefore \sqrt{a} + \sqrt{b} + \sqrt{c} = \sqrt{p}$.

8. Take any natural number. Multiply it with the next two natural numbers. Take another natural number different from the first and do the same as before. Subtract one result from the other to get a positive difference and divide the difference obtained by the positive difference of the original numbers. Add to the quotient the product of the original numbers. Prove that the final result is the product of some number by the number next above it.

Solution: Let $(x - 1)$ be the natural number. The next two natural numbers are x and $x + 1$.

Product $= (x - 1)x(x + 1)$.

Let $(y - 1)$ be another natural number $x \neq y$.

The next two natural numbers are y and $y + 1$.

Product $= (y - 1)y(y + 1)$.

Difference of the products

$$\begin{aligned} &= x(x^2-1) - y(y^2-1) \\ &= x^3 - y^3 - (x-y) \\ &= (x-y)(x^2+xy+y^2) - (x-y) \\ &= (x-y)(x^2+xy+y^2-1). \end{aligned}$$

Dividing this by the difference of the original numbers $(x-y)$, we get x^2+xy+y^2-1.

Let it be z.

$z = x^2+xy+y^2-1$

Add to this the product of the original numbers.

We get $x^2+xy+y^2-1+(x-1)(y-1)$

$$\begin{aligned} &= x^2+xy+y^2-1+xy-x-y+1 \\ &= x^2+2xy+y^2-x-y \\ &= (x+y)^2-(x+y) = (x+y)(x+y-1) \\ &= (x+y-1)(x+y) \\ &= \text{(some number) (the next number above it).} \end{aligned}$$

It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of the achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity.

Pierre-Simon de Laplace

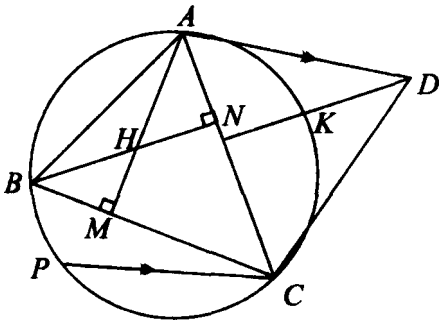
FINAL - BHASKARA CONTEST
(NMTC at JUNIOR LEVEL IX & X Standards)

1. H is the orthocenter of an acute angled triangle ABC, with circumcentre O. Let P be a point on the arc BC not containing A of the circumcircle different form B and C. Let D be a point such that $AD = PC$ and AD parallel to PC. Let K be the orthocentre of the triangle ACD. Prove that K lies on the circumcircle of triangle ABC.

Solution: Let K be the orthocentre of $\triangle ACD$. Since $AD||PC$ and $AD = PC$, $ADCP$ is a parallelogram

$\therefore \angle ADC = \angle CPA = \angle CBA = \angle B$.

In a $\triangle ABC$ if H is the orthocentre then $\angle BHC = \angle BHM + \angle CHM = 90° - \angle HBM + 90° - \angle HCM = \angle C + \angle B$.



(since $\angle HBM = \angle NBC = 90° - C$)

$= 180° - \angle A$.

$\therefore$ In $\triangle ADC$ $\angle AKC = 180° - \angle ADC = 180° - \angle B$

$\therefore \angle ABC + \angle AKC = 180°$

$\therefore$ K lies on the circum circle.

2. Find all positive integer solution of the equation $4x^3 - 3x - 1 = 2y^2$.

Solution:

$$\begin{aligned} 4x^3 - 3x - 1 &= (x-1)(4x^2 + 4x + 1) \\ &= (x-1)(2x+1)^2 = 2y^2 \end{aligned}$$

$2x + 1$ is odd. $\therefore$ $x - 1$ must be divisible by 2.

$\therefore$ $x - 1 = 2m$ say $m \geq 1$

$2m(2x+1)^2 = 2y^2$ $= m(2x+1)^2 = y^2$

$m(2(2m+1)+1)^2 = y^2$

$m(4m+3)^2 = y^2$

gcd $(m, 4m+3) = \gcd(m, 3)$

If gcd $(m, 3) = 1$, then both m and $(4m+3)^2$ must be perfect squares.

So we have infinitely many solutions $m = k^2$

$x = 2m + 1 = 2k^2 + 1 \qquad k = 0, 1, 2, 4, 5, 7, 8, \cdots$

and $y = k(8k^2 + 7)$.

If 3 divides m and gcd $(m, 3) = 3$ then

$$m = 3^k \cdot l \text{ where } k \geq 1 \text{ and } \gcd(3, l) = 1.$$

We get $3^k \cdot l(3(4 \cdot 3^{k-1} \cdot l + 1))^2 = y^2$

$$3^{k+2} \cdot l(4 \cdot 3^{k-1} \cdot l + 1))^2 = y^2$$

$\therefore$ k must be even and l a perfect square for a solution to exit. i.e., $3^{2k'} l'^2$ is a perfect square which was not already covered in the earlier case as we had gcd $(m, 3) = 1$.

Solutions are

$$\begin{aligned} (x, y) &= (2l^2 + 1, \quad k(8k^2 + 7)) \quad k \neq a \text{ multiply of } 3 \\ &= (2l'^2 \cdot 3^{2k'} + 1, \quad 3^{k'+1} l'(4 \cdot 3^{2k'-1} \cdot l'^2 + 1)) \end{aligned}$$

where $l' \geq 0$, $k' \geq 1$ integers.

3. Consider the set A of numbers $\left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{2012}\right\}$. We delete two of them say 'a' and 'b' and in their place we put only one number $a + b + ab$. After performing the operation 2011 times what is the number that is left over.

Solution: If α_1, α_2 are 2 real numbers then after such an operation the number $(\alpha_1 + 1)(\alpha_2 + 1) - 1$ remains.

If we have three numbers $\alpha_1, \alpha_2, \alpha_3$ without loss of generality we can say that the first operation is carried out between α_1, α_2 giving $(\alpha_1 + 1)(\alpha_2 + 1) - 1$ and α_3 as the remaining elements carrying out a second operation we get,

$((\alpha_1 + 1)(\alpha_2 + 1) - 1 + 1)(\alpha_3 + 1) - 1$

$= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) - 1$ as the number left.

Assuming that this is true for n numbers $\alpha_1, \cdots, \alpha_n$ carrying out $n-1$ such operations leads to the number $(\alpha_1+1)\cdots(\alpha_n+1) - 1$ for $n \leq m$, we need to prove it for $m + 1$ numbers $\alpha_1, \cdots, \alpha_m, \alpha_{m+1}$.

If the 1^{st} m numbers are reduced to one number after $m - 1$ reductions it will be $(\alpha_1+1)\cdots(\alpha_m+1)-1$. Now with α_{m+1}, the reductions will be $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_m+1)(\alpha_{m+1}+1)-1$.

If some k terms $k < m$ are reduced to 1 term and $m + 1 - k$ terms are reduced to one term after $k-1$ and $m+1-k-1, \cdots$ steps (a total of $m-1$ steps), we end up with $(\alpha_{i_1}+1)(\alpha_{i_2}+1)$ $\cdots(\alpha_{i_k}+1)-1$ and $(\alpha_{j_1}+1)\cdots(\alpha_{j_{m+1-k}}+1)-1$ when these are reduced. By the m^{th} step, we will get

$$\prod_{i=1}^{m+1}(\alpha_i + 1) - 1.$$

The above was the proof by induction for any n numbers $\alpha_1, \cdots, \alpha_n$.

Our numbers are $\frac{1}{1}, \frac{1}{2}, \cdots, \frac{1}{2012}$ for $n = 2012$. So the final number is

$$(1+1)(1+\tfrac{1}{2})(1+\tfrac{1}{3})+\cdots+(1+\tfrac{1}{2011})+(1+\tfrac{1}{2012})-1$$
$$=2\cdot\frac{3}{2}\cdot\frac{4}{3}+\cdots+\frac{2012}{2011}\cdot\frac{2013}{2012}-1$$
$$=2013-1=2012$$

4. Seven digit numbers are formed by the digits 1,2,3,4,5,6 and 7. In each number no digit is repeated. Prove that among all these numbers there is no number which is a multiple of another number.

Solution: Let there be two such 7 digit numbers a, b such that $a = bc$ where c is an integer greater than 1.

For both a, b the sum of the digits is $1+2+3+4+5+6+7 = 28$

and $28 \equiv 1(\text{mod}9)$

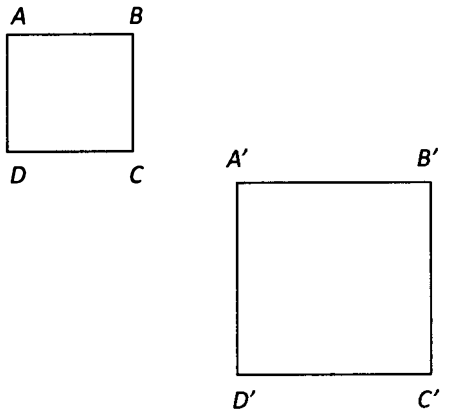
$\therefore \quad a \equiv b \equiv 1(\text{mod}9)$

Modulo 9 of $a = bc$ gives $c \equiv 1(\text{mod}9)$.

But $c > 1$. $\therefore\ c \geq 10$. $\therefore\ a \geq 10b > 10^7$

which is a contradiction since a, b are only 7 digit numbers.

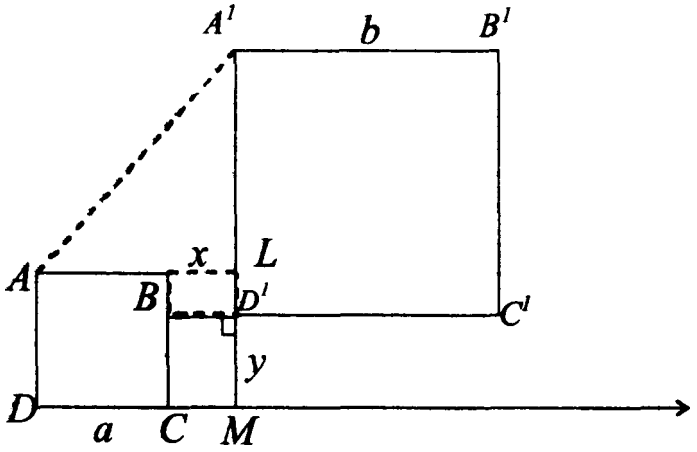
5. $ABCD$ and $A'B'C'D'$ are two unequal squares in a plane placed as in the figure ($A'B'$ parallel to AB etc.)



Prove that

$AA'^2+CC'^2+BB'^2+DD'^2$.

Solution: The square $A'B'C'D'$ is displaced from the square $ABCD$ with the corresponding sides parallel.



Let x be the displacement along (say x-direction) i.e., $BL = x$ and y be the displacement along the perpendicular direction (y-direction). i.e., $D'M = y$.

Let a, b be the sides respectively of the squares $ABCD$ and $A'B'C'D'$.

$$\text{Now } AA'^2 = (b+y-a)^2 + (a+x)^2, \qquad CC'^2 = y^2 + (b+x)^2$$
$$BB'^2 = (b+x)^2 + (b+y-a)^2, \qquad DD'^2 = y^2 + (a+x)^2$$

$$\begin{aligned} AA'^2 + CC'^2 &= (b-a+y)^2 + (a+x)^2 + y^2 + (b+x)^2 \\ BB'^2 + DD'^2 &= (b-a+y)^2 + (a+x)^2 + y^2 + (b+x)^2 \\ \Rightarrow\ AA'^2 + CC'^2 &= BB'^2 + DD'^2. \end{aligned}$$

6. Find integers x, y, z such that

$$x^2z + y^2z + 4xy = 40$$
$$x^2 + y^2 + xyz = 20$$

Solution: Let

$$x^2z + y^2z + 4xy = 40 \qquad (1)$$
$$x^2 + y^2 + xyz = 20 \qquad (2)$$

$(1) - 2 \times (2) \Rightarrow$

$x^2z + y^2z + 4xy - 2x^2 - 2y^2 - 2xyz = 0$

i.e., $(x^2z - 2x^2) + (y^2z - 2y^2) + (4xy - 2xyz) = 0$

$\Rightarrow \quad x^2(z-2) + y^2(z-2) - 2xy(z-2) = 0$

$\Rightarrow \quad (z-2)(x^2 + y^2 - 2xy) = 0$

$\Rightarrow \quad z = 2$ or $(x-y)^2 = 0$ (i.e.,) $x = y$

when $z = 2$ (1) $\Rightarrow (x+y)^2 = 20$

and x, y are integers and no square is 20.

$\therefore$ $z = 2$ cannot be a solution.

$\Rightarrow \quad x = y$

(2) $\Rightarrow \quad 2x^2 + zx^2 = 20$

$\Rightarrow \quad x^2(z+2) = 20$

$\Rightarrow \quad \frac{20}{z+2}$ must be a perfect square.

$\therefore$ $z + 2 = 20$ or $z + 2 = 5$

$\Rightarrow \quad z = 18$ or $z = 3$

$\Rightarrow \quad x = \pm 1$ or $x = \pm 2$.

The solutions are $(1,1,18)$, $(-1,-1,18)$, $(2,2,3)$ and $(-2,-2,3)$.

7. There are two natural numbers whose product is 192. It is given that the quotient of the arithmetic mean to the harmonic mean of their greatest common measure and the least common multiple is $\frac{169}{48}$. Find the numbers.

Solution: Let x, y be the two numbers.

Given: $xy = 192$ (1)

Let p be the GCM and q the LCM of x and y.

Given: $\frac{\frac{p+q}{2}}{\frac{2pq}{p+q}} = \frac{169}{48} \Rightarrow \frac{(p+q)^2}{4pq} = \frac{169}{48}$.

(Now $pq = xy = 192$) $\therefore$ $(p+q)^2 = \frac{169}{48} \times 4 \times 192$.

$\Rightarrow \quad p+q=52$ and $pq=192$.

Solving we get $p=4,\ q=48$.

Let $x=4l,\ y=4m$. $\because \quad xy=192$.

$16lm=192 \quad \Rightarrow lm=12$.

Prime factors of 12 are (3,4) and even though 1 is not a prime; (1,12) also works.

$x=4\times 3=12 \qquad y=4\times 4=16$

$x=4\times 1=4 \qquad y=4\times 12=48$

The numbers are 12,16 or 4,48.

8. Find all the positive integral solution of the equation

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{2013}.$$

Solution: x, y must be > 2013.

We need to get y in terms of x

$$\frac{1}{y}=\frac{1}{2013}-\frac{1}{x}=\frac{x-2013}{2013\cdot x}$$

$$y=\frac{2013\cdot x}{x-2013}=\frac{2013(x-2013)+2013^2}{x-2013}$$

$$=2013+\frac{2013^2}{x-2013}$$

This means $x-2013$ must divide 2013^2 exactly

$2013^2=3^2\times 11^2\times 61^2$

$x-2013$ must be a factor of $3^2\times 11^2\times 61^2$

$x-2013=1,3,11,61,\ 3\times 11,11\times 61,3\times 61,3\times 11\times 61$

$3^2,11^2,61^2,3^2\times 11,3^2\times 61,3^2\times 11\times 61$

$11^2\times 3,11^2\times 61,11^2\times 61\times 3,61^2\times 3,61^2\times 11$

$61^2\times 11^2\times 3,3^2\times 11^2,3^2\times 11^2\times 61,11^2\times 61^2$

$11^2\times 61^2\times 3,61^2\times 3^2,61^2\times 3^2\times 11,3^2\times 11^2\times 61^2$.

27 solutions for x and the corresponding values for y.

FINAL - RAMANUJAN CONTEST
(NMTC at INTER LEVEL XI & XII Standards)

1. Find all the pairs (x, y) where x, y are integers satisfying $(2x-1)^3 + 16 = y^4$.

Solution:

If $x \leq -1$ then $(2x-1)^3 \leq -27$ making the LHS negative. But RHS is positive. Hence no solutions for $x \leq -1$. $x = 0$ also there is no solution.

Note that LHS is odd. $\therefore$ RHS is also odd, thus making y odd and $y > 1$.

$$\begin{aligned}(2x-1)^3 &= y^4 - 16 = (y^2-4)(y^2+4) \\ &= (y-2)(y+2)(y^2+4)\end{aligned}$$

As y is odd, $y-2$, $y+2, y^2+4$ are all odd and pairwise relatively prime.

$$\begin{aligned}(y-2, y+2) &= (y-2, 4) = 1 \text{ as } y-2 \text{ is odd} \\ (y-2, y^2+4) &= (y-2, 8) = 1 \text{ as } y-2 \text{ is odd.}\end{aligned}$$

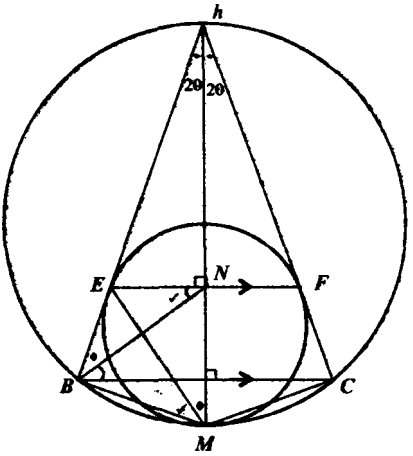
Each of them must be a perfect cube (i.e.,) $y-2 = a^3$ $y+2 = b^3$ and $y^2+4 = c^3$.

But as the difference between successive perfect cubes cannot be 4 and $y+2$ cannot be simultaneously a^3, b^3, there is no solution for this Diophantine equation.

2. I is the incentre of the isosceles triangle ABC in which $AB = AC$. Let Σ be a circle which touches AB at E and AC at F and touches the circumcircle of triangle ABC internally. Prove that I lies on EF.

Solution: Let M be the mid point of the arc BC not containing A. Since $AB = AC$, AM is the diameter of the

circumcircle and it is $\perp$r to BC.



Let $\angle BAM = \angle MAC = 2\theta$

Since $AE = AF$, $\triangle AEF$ is isosceles and $EF \| BC$.

$\therefore$ AM is $\perp$ EF meeting EF at N.

Claim: N is the incentre.

$\angle ABM = \angle ACM = 90°$ ($\angle$ in a semicircle)

$\therefore$ $ENMB$ is cyclic.

$\angle AEN = \angle AMB = 90 - 2\theta$

$\angle MBC = \angle MCB = \angle MAC = 2\theta$
($MB = MC$ chords subtending equal angles)

$\angle AEF = \angle EMF$ (angle in alternate segment)

$= 90 - 2\theta$

$\triangle ENM \equiv \triangle FNM$. ($EN = NF$ and NM common in right $\triangle$s)

$EM = FN$

$\therefore$ $\angle EMN = \frac{90-2\theta}{2} = 45 - \theta$.

$\therefore$ $\angle EMB = 90 - 2\theta - (45 - \theta) = 45 - \theta$.

As $ENMB$ is cyclic, $\angle ENB = \angle EMB = 45 - \theta$

$= \angle NBC \quad (\because \ EN || BC)$

$\therefore \ \angle EBN = \angle EMN = 45 - \theta.$

$\therefore \ \angle EBN = \angle NBC.$

BN bisects $\angle ABC$. Also AN bisects $\angle BAC$ and N is the incentre.

3. A function $f : Q \to Q$, where is the set of rational numbers, satisfies the conditions

(a) $f(1) = 2.$

(b) $f(xy) + f(x + y) = f(x) \cdot f(y) + 1$ for all $x, y \in Q.$ Determine all such functions f, with proof.

Solution: Put $x = 0, y = 1$ in

$$f(xy) + f(x + y) = f(x) \cdot f(y) + 1 \qquad (I)$$

$f(0) + f(1) = f(0) \cdot f(1) + 1$

$f(0) + 2 = 2f(0) + 1.$

$\therefore \ f(0) = 1$

Put $x = y = 1$ in (I)

$f(1) + f(2) = f(1) \cdot f(1) + 1.$

$\therefore \ f(2) = 3$

Put $y = 1$ in (I)

$f(x) + f(x + 1) = f(x) \cdot f(1) + 1$

$= 2f(x) + 1$

$$\therefore \ f(x + 1) = f(x) + 1 \qquad (II)$$

By induction

$f(n) = n + 1$ for all $n \in \mathbb{N}$

$f(0) = f(-1) + 1$. $\quad \therefore \quad f(-1) = 0$.

Again using induction from (II) and $f(-1) = 0$, we get,

$f(-n) = -n + 1$ for all $n \in \mathbb{N}$

i.e., $f(x) = x + 1$ if $x \in \mathbb{Z}$.

From (II), by induction, we get

$f(x + n) = f(x) + n$ for $x \in \mathbb{Q}$ and $n \in \mathbb{N}$.

If we take $x + n = y$ and $x = y - n$, we get

$f(y) = f(y - n) + n$

i.e., $f(y + (-n)) = f(y) + (-n)$

Thus we get

$$f(x + n) = f(x) + n \text{ for all } x \in \mathbb{Q} \text{ and } n \in \mathbb{Z}. \qquad (III)$$

Put $y = n$ in (I). We get

$f(nx) = f(n+1)f(x) - f(x+n) + 1 = f(n+1)f(x) - f(x) - n + 1$

$$f(nx) = nf(x) - n + 1 \quad \text{(True for } n \in \mathbb{Z}) \qquad (IV)$$

In (IV) put $n = q$ and $x = \frac{1}{q}$ where $q \in \mathbb{N}$

$f(1) = (q + 1)f(\frac{1}{q}) - q + 1$

$2 - 1 + q = (q + 1)f(\frac{1}{q})$

$\therefore \quad f(\frac{1}{q}) = \frac{q+1}{q} = (1 + \frac{1}{q})$

Consider $x = \frac{p}{q}$, $\quad p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

$f(\frac{p}{q}) = pf(\frac{1}{q}) - p + 1$

$= p(1 + \frac{1}{q}) - p + 1$

$= \frac{p}{q} + 1$.

$\therefore \quad f(x) = x + 1$ for all $x \in \mathbb{Q}$.

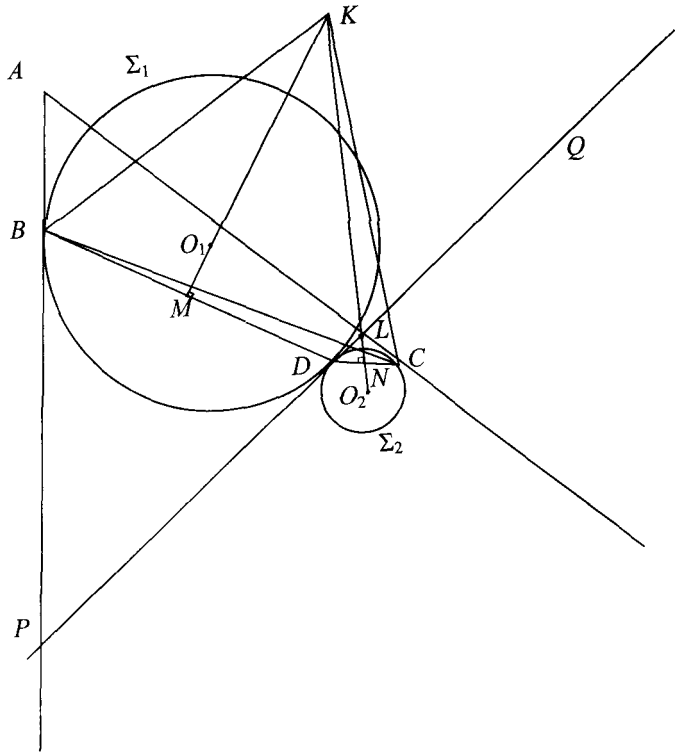
4. Let B be a point on the circle Σ_1 and A be a point on the tangent at B to $\Sigma_1 (B \neq A)$. Let C be a point not on Σ_1

such that AC meets Σ_1 in two distinct points. Let Σ_2 be a circle touching AC at C and Σ_1 at D on the same side of AC as B. Prove that the circumcentre of triangle BCD lies on the circumcircle of triangle ABC.

Solution: Let K be point of intersection of the $\perp$r bisectors of CD and BD, whose midpoints are N and M respectively.

$$\angle ADB = \angle QDB \qquad (I)$$

as AB and QD A are tangents to Σ_1.



Note that QD and AC are tangents to Σ_2 and they meet at L. Since $LD = LC$. L lies on the $\perp$r bisectors of CD i.e., L lies on KN.

$$\therefore \quad \angle QDC = \angle ACD \qquad (II)$$

Since K is the circumcentre of $\triangle BDC$,

reflex $\angle BKC = 2\angle BDC$

$$\begin{aligned}
\angle BKC &= 360 - 2\angle BDC \\
&= \angle ABD + \angle BDC + \angle DCA + \angle CAB - 2\angle BDC \\
&= (\angle ABD + \angle DCA) + \angle BAC - \angle BDC \\
&= (\angle QDB + \angle DCL) + \angle BAC - \angle BDC \text{ by (I)} \\
&= (\angle QDB + \angle QDC) - \angle BDC + \angle BAC \\
&= \angle BDC - \angle BDC + \angle BAC \\
&= \angle BAC.
\end{aligned}$$

K lies on the circum circle of $\triangle ABC$.

5. Find all positive integers x, y, z such that $8^x + 15^y = 17^z$.

Solution: The only solution is $x = y = z = 2$.

Let $8^x + 15^y = 17^z$ be equation (*).

Take modulo 4 of (*), we get

$(-1)^y \equiv 1(\text{mod}4)$

$\Rightarrow$ y must be even.

Take modulo 16 of (*), we get

$8^x + (-1)^y \equiv 1(\text{mod}16)$

Since y is even $8^x \equiv 0(\text{mod}16)$ $\Rightarrow$ $x \geq 2$

We have $17^2 \equiv 1(\text{mod}32)$ and $15^2 \equiv 1(\text{mod}32)$

$\Rightarrow$ If z is odd, then since $2|y$ and $x \geq 2$ then

$1 \equiv 17(\text{mod}32)$ (using (*)) which is a contradiction.

$\therefore$ z must be even.

Let $y = 2m$ and $z = 2n$ for some integers m, n. (*) can be factorized as $(17^n - 15^m)(17^n + 15^m) = 8^x$. Clearly, the gcd of the two divisors of LHS of this equation is 2 but the RHS is a power of 2.

$17^n - 15^m = 2 \quad (1), \qquad 17^n + 15^m = 2^{3x-1} \quad (2)$.

Taking modulo (32) of (1), we get, m, n must be odd.

(Otherwise the LHS of (1) will be 0, 14, 16 (mod 32)).

Adding (1) and (2), we get $17^n = 1 + 2^{3x-2}$.

If $x \geq 3$, RHS is 1(mod 32). But as n is odd, LHS is $17 \equiv 1$(mod 32) which is impossible. Thus $x = 2 \Rightarrow n = 1$. $\therefore z = 2$ and $m = 1$. $\therefore y = 2$.

6. A Pythagorean triangle is a right triangle in which all the three sides are of integer lengths. Let a, b be the legs of a Pythagorean triangle, and h be altitude to the hypotenuse. Determine all such a triangles for which $\frac{1}{a} + \frac{1}{b} + \frac{1}{h} = 1$.

Solution: Let C be the length of the hypotenuse.

Let A be the area of the triangle.

Clearly, $A = \frac{1}{2}ab = \frac{1}{2}ch$

$$\Rightarrow \quad ab = ch \qquad (1)$$

$$\text{Now} \quad 1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{h} = \frac{bh + ah + ab}{abh}$$

$$= \frac{bh + ah + ch}{abh} \quad \text{(using (1))}$$

$$= \frac{h(a + b + c)}{abh} = \frac{a + b + c}{ab}$$

$$\Rightarrow \quad a + b + c = ab \qquad (2)$$

We have $c^2 = a^2 + b^2$

(2) gives $ab - a - b = \sqrt{a^2 + b^2}$

$$\Rightarrow \quad (ab - a - b)^2 = a^2 + b^2$$

$$\Rightarrow \quad a^2b^2 + a^2 + b^2 - 2a^2b + 2ab - 2b^2a = a^2 + b^2$$

$$\Rightarrow \quad a^2b^2 - 2a^2b + 2ab - 2b^2a = 0$$

$$\Rightarrow \quad ab(ab - 2a - 2b + 2) = 0$$

$$\begin{aligned} \text{Since} \quad & ab \neq 0,\ ab - 2a - 2b + 2 = 0 \\ \Rightarrow \quad & a(b-2) = 2b - 2 \\ \Rightarrow \quad & a = \frac{2(b-1)}{b-2} = \frac{2(b-2)+2}{b-2} \\ \Rightarrow \quad & a = 2 + \frac{2}{b-2}. \end{aligned}$$

Since a is an integer, $b-2 = 1$ or 2.

If $b = 3$ then $a = 4$ and $c = 5$

$b = 4$ then $a = 3$ and $c = 5$

The only triangle satisfying the condition $\frac{1}{a} + \frac{1}{b} + \frac{1}{h} = 1$ is the $(3,4,5)$ triangle.

7. Let $f(n)$ be a function defined on the non-negative integer n. Given

(a) $f(0) = f(1) = 0$

(b) $f(2) = 1$

(c) for $n > 2, f(n)$ gives the smallest positive integer which does not divide n.

Let $g(n) = f(f(f(n)))$.

Find the value of $S_{2012} = g(1) + g(2) + \cdots + g(2012)$.

Solution:

$g(1) = f(f(f(1))) = f(f(0)) = f(0) = 0$ and

$g(2) = f(f(f(2))) = f(f(1)) = f(0) = 0$.

Let n be an odd integer then

$g(n) = f(f(f(n))) = f(f(2)) = f(1) = 0$.

$\because$ The smallest positive integer which is 2.

$$\begin{aligned} \therefore \quad S_{2012} &= g(1) + g(2) + g(3) + \cdots + g(2012) \\ &= g(4) + g(6) + g(8) + \cdots + g(2012) \end{aligned}$$

Consider the set $S = \{4, 6, 8, 10, 12, \cdots, 2012\}$

There are 1005 terms in this set, all of them are even.

Consider the subset of S (say S_1) which consists of elements which are not multiplies of 3. [i.e, of the form $3m \pm 1$]. S_2 consists of elements which are multiplies of 3.

$S_2\{6, 12, 18, \cdots, 2010\}$

$n(S_2) = 335 \quad \{6 + (n-1)6 = 2010 \quad \Rightarrow \quad n = 335\}$.

$\therefore \quad n_1(S_1) = 1005 - 335 = 770$.

Consider $x \in S_1$. $\quad \therefore \quad x = 3m \pm 1$.

$g(x) = f(f(f(x))) = f(f(f(3m \pm 1))) = f(f(3))$

$= f(2) = 1$.

Thus $g(x) = 1$ for all $x \in S_1$.

$\therefore \quad \Sigma g(x) = 770$ where $x \in S_1$.

Let us now take S_2.

$g(6) = f(f(f(6))) = f(f(4)) = f(3) = 2$

$g(12) = f(f(f(12))) = f(f(5)) = f(2) = 1$

$g(18) = f(f(f(18))) = f(f(4)) = f(3) = 2$

It is clear that $g(6p) = 2$ for all odd numbers p
and $g(6q) = 1$ for all even numbers q.

Thus the values of $g(x)$, $x \in S_2$ are

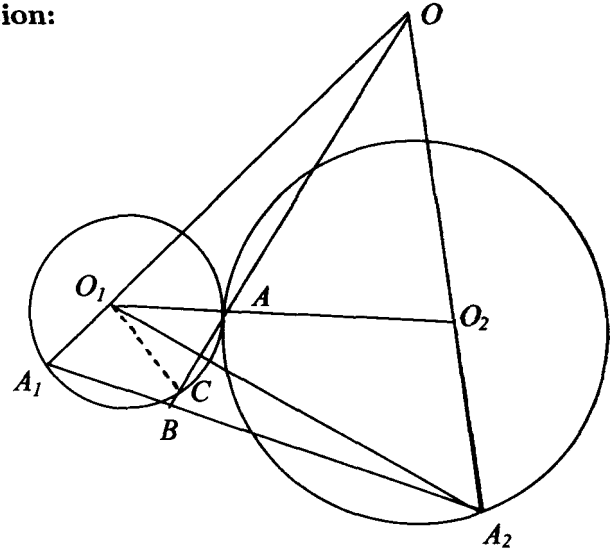
2,1,2,1, $\cdots$ 335 terms.

Thus there are 168 twos and 167 ones.

$$\begin{aligned} \Sigma g(x) &= 2 \times 168 + 1 \times 167 \\ &= 336 + 167 = 503. \\ \therefore \quad S_{2012} &= 770 + 503 = 1173. \end{aligned}$$

8. In a circle C with centre O and radius r, let C_1 and C_2 be two circles with centres O_1, O_2 and radii r_1 and r_2 respectively be situated such that each circle C_1 and C_2 is internally tangent to C at A_1 and A_2 respectively and such that C_1 and C_2 are externally tangent to each other at A. Prove that the three lines OA, O_1A_2 and O_2A_1 are concurrent.

Solution:



$$\frac{A_1B}{\sin A_1\hat{O}B} = \frac{OB}{\sin O\widehat{A_1}B}$$

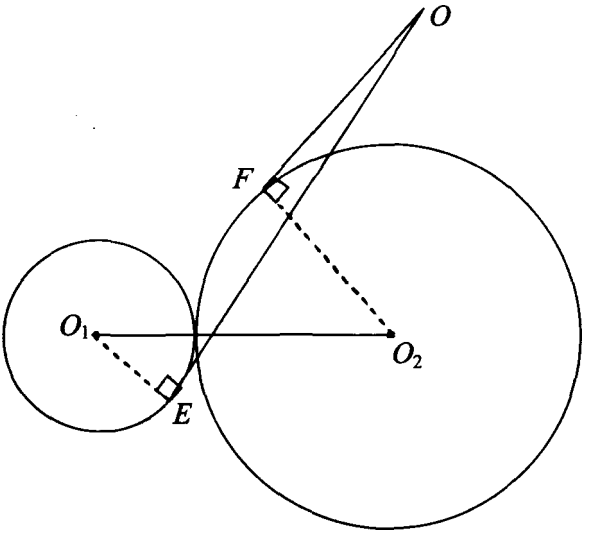
$$\frac{OB}{\sin O\widehat{A_2}B} = \frac{A_2B}{\sin A_2\hat{O}B}$$

(But $OA_1 = OA_2 =$ radius of the biggest circle touching both the smaller circles $\Rightarrow O\widehat{A_1}B = O\widehat{A_2}B$).

$$\therefore \quad \frac{A_1B}{\sin A_1\hat{O}B} = \frac{OB}{\sin O\widehat{A_1}B} = \frac{OB}{\sin O\widehat{A_2}B} = \frac{A_2B}{\sin A_2\hat{O}B}$$

$$\Rightarrow \quad \frac{A_1B}{A_2B} = \frac{\sin A_1\hat{O}B}{\sin A_2\hat{O}B} = \frac{O_1G}{O_1O} \Big/ \frac{O_2H}{O_2O} = \frac{O_1G}{O_1O} \cdot \frac{O_2O}{O_2H}$$

(where O_1G and O_2H are drawn perpendiculars from O_1 and O_2 to OB which are not shown).



The triangles GO_1A and HO_2A are obviously similar

$$\Rightarrow \quad \frac{O_1G}{O_1A} = \frac{O_2H}{O_2A} \text{ or } \frac{O_1G}{O_2H} = \frac{r_1}{r_2}$$

$$\frac{A_1B}{A_2B} = \frac{r_1}{r_2} \cdot \frac{O_2O}{O_1O} = \frac{r_1}{r_2} \cdot \frac{r - r_2}{r - r_1}$$

i.e., $\dfrac{A_1B}{A_2B} \cdot \dfrac{r_2}{r - r_2} \cdot \dfrac{r - r_1}{r_1} = 1$.

$\Rightarrow$ by converse of Ceva's Theorem, the lines OA, O_1A_2 and O_2A_1 are concurrent.

No human investigation can be called real science if it cannot be demonstrated mathematically

Leonardo da Vinci

ARYABHATA CONTEST - SENIOR

1. Let $P > 2$ a prime number and let $a_k \in \{0, 1, 2, \cdots, p^2 - 1\}$ denote the value of k^p modulo p^2. Find $\sum_{k=1}^{p-1} a_k$.

Solution: We have $(p-k)^p = (-k)^p + pC_1(-k)^{p-1}p + pC_2(-k)^{p-1}p^2 + \cdots + p^p \equiv -k^p (\mathrm{mod} p^2)$. Therefore,

$$a_{p-k} = p^2 - a_k \tag{1}$$

Summing up these equalities, we obtain

$$\sum_{p=1}^{k-1} a_{p-k} = p^2(p-1) - \sum_{k=1}^{p-1} a_k$$

But

$$\sum_{p=1}^{k-1} a_{p-k} = \sum_{k=1}^{p-1} a_k$$

$$\therefore \sum_{k=1}^{p-1} a_k = p^2(p-1) - \sum_{k=1}^{p-1} a_k \Rightarrow \sum_{k=1}^{p-1} a_k = \frac{p^2(p-1)}{2}$$

Thus $\boxed{\sum_{k=1}^{p-1} a_k = \frac{p^2(p-1)}{2}}$

2. If a, b are rational numbers satisfying $\tan a\pi = b$, then find all possible values of b.

Solution: Let $a = \frac{m}{n}$, where gcd $(m, n) = 1$. By De-Moivre's Theorem, we get $(\cos a\pi + i \sin a\pi)^n = \cos(an)\pi + i\sin(an)\pi = \cos m\pi + i \sin m\pi$.

Similarly, $(\cos a\pi - i \sin a\pi)^n = \cos m\pi - i \sin m\pi$

$$\therefore \left(\frac{1+ib}{1-ib}\right) = \left(\frac{1 + i\tan a\pi}{1 - i\tan a\pi}\right)^n = \left(\frac{e^{ia\pi}}{e^{-ia\pi}}\right)^n$$

$$= \left(e^{i2a\pi}\right)^n = e^{i2(an)\pi} = e^{i2m\pi} = 1.$$

Now $\dfrac{1+ib}{1-ib} = \dfrac{1-b^2}{1+b^2} + i\dfrac{2b}{1+b^2} \in z[i]$, where $z[i]$ is set of all $x+iy$, such that x, y are integers.

Hence $\dfrac{1-b^2}{1+b^2} \in z$, $\dfrac{2b}{1+b^2} \in z \Rightarrow 1-b^2 = 0$ and $2b = 0$

$\therefore$ $b = -1, 0, 1$ are the only possible values of b.

3. Find the minimum number of elements that must be deleted from the set $\{1, 2, \cdots, 2012\}$ such that the set of remaining elements does not contain two elements together with their product.

   Solution: If we extract the numbers $1, 2, 3, \cdots, 44$ and then choose $x, y \in \{45, 46, \cdots, 2012\}$ then $xy \geq 45^2 > 2012$. Thus such x, y cannot belong to the given set.

   $\therefore$ The number of elements to be deleted is atmost 43.

   Consider now the triples (2,87,2.87), (3,86,3.86), $\cdots$, (44,45,44.45) we must delete at least one number from each triple in order that the remaining set be free of products. Thus the minimum number of elements to be deleted is 43.

4. Define the sequence T_n by $T_1 = 2, T_{n+1} = T_n^2 - T_n + 1$, for $n \geq 1$. If $m \neq n$ find the greatest common divisor of T_m, T_n and find $\sum_{i=1}^{\infty} \dfrac{1}{T_i}$.

   Solution: $T_{n+1} = T_n^2 - T_n + 1$ and so $T_n = T_{n-1}^2 - T_{n-1} + 1$.

   Now $T_{n+1} = T_n^2 - T_n + 1 = T_n(T_n - 1) + 1 = T_n(T_{n-1}^2 - T_{n-1}) + 1$

   $T_{n+1} = T_n \cdot .T_{n-1}(T_{n-1} - 1) + 1 = T_n \cdot T_{n-1} \cdots T_1 + 1$ and so

   $T_{n+1} = 1 + T_1 T_2 \cdots T_n$.

   Let $m < n$. Then T_m divides $T_1 T_2 \cdots T_{n-1} = T_n - 1$ and thus g.c.d of T_m, T_n is 1.

   (i.e.) g.c.d. $(T_m, T_n) = 1 \Rightarrow T_m, T_n$ are relatively prime if $m < n$.

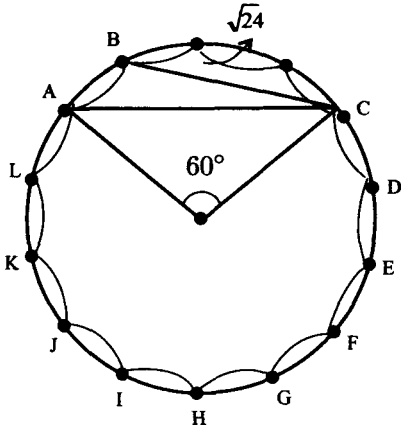
Now, $\sum_{i=1}^{k+1} \frac{1}{T_i} = 1 - \frac{1}{T_{k+1} - 1} + \frac{1}{T_{k+1}}$

$$= 1 - \frac{1}{T_{k+1}(T_{k+1} - 1)} = 1 - \frac{1}{T_{k+2} - 1}.$$

Since T_n tends to infinity, $\sum_{i=1}^{\infty} \frac{1}{T_i} = 1 - 0 = 1.$

5. A convex polygon of twelve sides inscribed in a circle has in some order six sides of length $\sqrt{2}$ and six of length $p\sqrt{24}$. Find he radius of the circle.

Solution: Let AB, BC be a pair of adjacent but unequal sides. Then arc AB+ arc BC = arc AC = $\frac{360°}{6}$ = 60°, so $AC = r$, where r is the radius of the circle.



Applying law of cosines to triangle ABC, we get

$(AC)^2 = r^2 = (AB)^2 + (BC)^2 - 2(AB) \cdot (BC)cos(\angle ABC)$.

Now $\angle ABC = (\frac{10}{12})\pi = \frac{5\pi}{6} = 150°$.

$\therefore \quad r^2 = (\sqrt{2})^2 + (\sqrt{24})^2 - 2(\sqrt{2})(\sqrt{24})\cos(150°)$

$r^2 = 2 + 24 - 2\sqrt{48}(\frac{-\sqrt{3}}{2}) = 26 + 12 = 38.$

$\therefore r = \sqrt{38}$ is the radius of the circle.

6. For sequences $\{A_n\}$ and $\{B_n\}$ of events on some sample space S, let $\lim_{n\to\infty} P(B_n) = 1$.

Determine $\lim_{n\to\infty} [P(A_n/B_n) - P(A_n)]$, where $P(A/B)$, is the conditional probability of A given B.

Solution: Since $\lim_{n\to\infty} P(B_n) = 1$ (1), $\lim_{n\to\infty} [1 - P(B_n)] = 0$ (2)

Further, $P(A_n \cup B_n) \geq P(B_n) \Rightarrow \lim_{n\to\infty} P(A_n \cup B_n) = 1$ (3)

Now, $P\left(\frac{A_n}{B_n}\right) - P(A_n) = \frac{P(A_n \cap B_n)}{P(B_n)} - P(A_n)$

$$= \frac{P(A_n \cap B_n) - P(A_n).P(B_n)}{P(B_n)}$$

$$= \frac{P(A_n) + P(B_n) - P(A_n \cup B_n) - P(A_n).P(B_n)}{P(B_n)}$$

$$= \frac{P(A_n)[1 - P(B_n)] + P(B_n) - P(A_n \cup B_n)}{P(B_n)}$$

$\therefore \lim_{n\to\infty} \left[P\left(\frac{A_n}{B_n}\right) - P(A_n)\right] =$

$$\frac{\lim_{n\to\infty} \{(P(A_n))[1 - P(B_n)] + P(B_n) - P(A_n \cup B_n)\}}{\lim_{n\to\infty} P(B_n)}$$

$$= \frac{0 + 1 - 1}{1} = 0 \quad \text{[From equations 1,2 and 3].}$$

REGIONAL MATHEMATICAL OLYMPIAD - 2012

1. The incircle of $\triangle ABC$ divides the median AA', where A' is the midpoint of BC, into three equal parts. If the side $BC = 4$ units, then find the other two sides.

Solution: Let D, E be the points of contact of the incircle with BC and CA respectively. Let AA' cut the circle at M, N. Then we have

$$AE^2 = AM \cdot AN.$$

Suppose $AM = x = MN = NA'$. Then we have

$$\begin{aligned}
AE^2 = (s-a)^2 &= 2x^2 \\
\left(\frac{b+c-a}{2}\right)^2 &= 2x^2 \\
2x^2 = A'N \cdot A'M &= A'D^2 \\
A'D^2 &= \left(\frac{a}{2} - BD\right)^2 \\
2x^2 &= \left(\frac{a}{2} - (s-b)\right)^2 \\
2x^2 &= \left(\frac{b-c}{2}\right)^2 \\
(b+c-a)^2 &= (b-c)^2 \\
(a-2b)(a-2c) &= 0.
\end{aligned}$$

This means that $a = 2b$ or $a = 2c$. If $a = 2b$, then $b = 2$ units as $a = BC = 4$ units. Now

$$AA'^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4} = 9x^2 = \frac{9}{2}\frac{(b-c)^2}{4}$$

This gives

$$13b^2 + 5c^2 - 18bc = (b - c)(13b - 5c) = 0.$$

Hence either $b = c$ or $13b = 5c$. But $b = c$ is not possible as that would make D and A' coincide. As $b = 2$ we get

$c = \frac{26}{5}$. Thus we have $a : b : c = 10 : 5 : 13$.

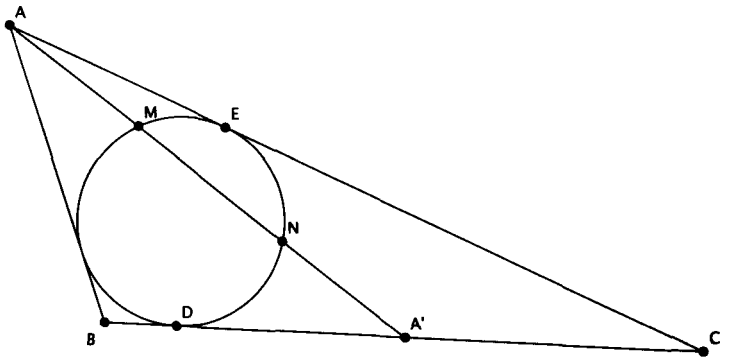


Figure 1:

2. Let a be an integer. Consider the function

$$y = \frac{12x^2 - 12ax}{x^2 + 36}$$

For what integral values of a the maximum and the minimum of the function $y = y(x)$ are integers?

Solution: We have

$$(y - 12)x^2 + 12ax + 36y = 0.$$

This quadratic equation in x has a solution in real numbers if and only if the discriminant

$$D = 12^2a^2 - 4 \cdot 36y(y - 12) = 12^2(a^2 - y^2 + 12y) \geq 0.$$

This happens if and only if $y^2 - 12y - a^2 \leq 0$ and this holds if and only if

$$6 - \sqrt{36 + a^2} \leq y \leq 6 + \sqrt{36 + a^2}.$$

The end points of the above interval are the minimum and maximum of y. If we want them to be integers then $36 + a^2 =$

b^2 must be a perfect square. This gives $36 = (b + a)(b - a)$. If we take $36 = mn$ as any factorization of 36 into positive integers, then $a = \frac{1}{2}(m - n)$ and $b = \frac{1}{2}(m + n)$. Now a, b are integers implies that m, n are both odd or they are both even. This gives two possibilities

$$36 = 6 \times 6 \text{ or } 36 = 2 \times 18.$$

Now, $m = 6 = n$ gives $a = 0$ and this forces

$$x^2 = -\frac{36y}{y - 12}$$

and this has a solution if and only if $0 \leq y < 12$. As we are interested in the maximum and minimum $a \neq 0$ and we have the corresponding factorization $36 = 2 \times 18$ which gives $a = \pm 8$ and $b = 10$. So there are two integers $a = 8$ and $a = -8$ with the required properties.

3. Denote by $d(k)$ the number of positive divisors of a natural number k. For example $d(6) = 4$ and $d(7) = 2$. The function d has domain $\mathbb{N}$ and $d(x)$ is defined if and only if $x \in \mathbb{N}$. Suppose n is a solution to the equation

$$d\left(\frac{8}{5}n\right) = \frac{8}{5}(n)$$

then find the value of the ratio $d\left(\frac{4}{25}n\right) : d(n)$.

Solution: As $d(k)$ is defined only when k is a positive integer it is evident that $\frac{8}{5}n$ must be a natural number, and hence we must have

$$n = 2^m 5^p n', \text{ with } p \geq 1$$

for some natural number n' that is coprime to 10. In other words, $gcd(n', 10) = 1$. If $k = \prod_{i=1}^{q} p_i^{s_1}$ is the prime factorization of $k \in \mathbb{N}$ then any positive divisor j of k is of

the form $j = \prod_{i=1}^{q} p_i^{t_i}$ with $0 \leq t_i \leq s_i$ for $i = 1, 2, \cdots, q$. This means that $d(k) = \prod_{i=1}^{q} (s_i + 1)$. Thus the given equation

$$d\left(\frac{8}{5}n\right) = \frac{8}{5}d(n)$$

implies that

$$(m+4)pd(n') = \frac{8}{5}d(n) = \frac{8}{5}(m+1)(p+1)d(n').$$

Therefore

$$5(m+4)p = 8(m+1)(p+1)$$

or

$$3mp - 12p + 8m + 8 = 0$$

or

$$(3p+8)(m-4) + 40 = 0$$

or

$$(3p+8)(4-m) = 40.$$

Now, $3p + 8 \geq 11$ since $p \geq 1$; and $3p + 8 = 2 \mod 3$. So, the only such possible factorization of 40 satisfies

$$3p + 8 = 20 \quad \text{and } 4 - m = 2.$$

This gives $p = 4$ and $m = 2$ and hence $n = 2^2 5^4 n'$. We want the ratio $d(\frac{4}{25}n) : d(n)$. We have

$$d\left(\frac{4}{25}n\right) = 5 \times 3d(n')$$

Also $d(n) = (m+1)(p+1)d(n') = 3 \times 5d(n')$. This gives the required ratio as

$$\frac{d(\frac{4}{25}n)}{d(n)} = \frac{5 \times 3}{3 \times 5} = 1.$$

4. Let I be the incenter of $\triangle ABC$ and let X, Y, Z be the feet of the perpendiculars from I on the sides BC, CA, AB respectively. If IX meets YZ at N, then prove that A, N and the midpoint A' of BC are collinear.

Solution:

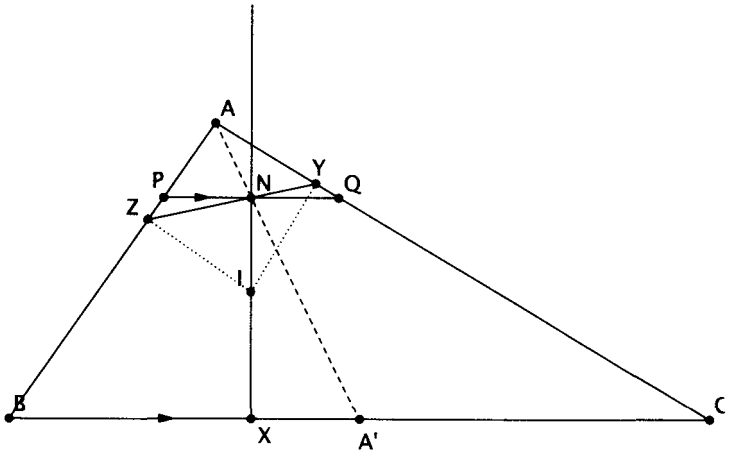


Figure 2:

Draw the parallel through N to BC meeting AB, AC at P, Q respectively. We have $NX \perp PQ$. The quadrilateral $INPZ$ is cyclic since $\angle INP = 90° = \angle IZP$. Similarly $INYQ$ is also cyclic. Therefore we have

$$\angle PNZ = \angle PIZ \quad \text{and } \angle YNQ = \angle YIQ.$$

But the vertically opposite angles $\angle PNZ = \angle YNQ$ and therefore $\angle PIZ = \angle QIY$. This means that the two right triangles $\triangle PIZ$ and $\triangle QIY$ are congruent since we have $IZ = IY = inradius$. This means that $IP = IQ$. Now, $IN \perp PQ$ implies that N is the mid-point of PQ. This along with the fact that $PQ \| BC$ implies that A, N, A' are collinear.

5. For arbitrary positive numbers a, b, c prove that

$$\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b} \geq 1.$$

Solution: Let $x = b + 2c,\ y = c + 2a,\ z = a + 2b$. This gives

$$a = \frac{1}{9}(-2x + 4y + z),\ b = \frac{1}{9}(x - 2y + 4z),\ c = \frac{1}{9}(4x + y - 2z).$$

We have to prove the inequality

$$\frac{-2x + 4y + z}{9x} + \frac{x - 2y + 4z}{9y} + \frac{4x + y - 2z}{9z} \geq 1.$$

This can be rewritten as

$$\left(\frac{x}{y} + \frac{y}{x}\right) + \left(\frac{y}{z} + \frac{z}{y}\right) + \left(\frac{z}{x} + \frac{x}{z}\right) + 3\left(\frac{y}{z} + \frac{x}{z} = \frac{z}{y}\right) \geq 15.$$

Each of the expressions in the first three parenthesis is greater than equal to 2 and the expression in the third parenthesis is at least 3 by AM-GM inequality. Hence the left side is at lease $(3 \times 2) + (3 \times 3) = 15)$ and we are done.

6. A private club has 90 rooms and 100 members. Keys must be given to the members such that the following conditions hold:

   (a) Each set of 90 members can be assigned to 90 distinct rooms whose doors they can open.

   (b) Each key opens only one room.

   (c) A single member may be given several keys.

   (d) A single room can have several keys.

   The management wants to minimize the total number of keys. Find the minimum number of keys and prove that smaller number of keys will not work.

   Solution: Suppose each of the first 90 members is given one key for the 90 different rooms. Also, each of the remaining 10 members is given 90 keys to the 90 different rooms. This makes it 990 keys in all. Clearly this gives a solution satisfying the requirements.

Claim: The minimum number of keys required, satisfying the given conditions is 990.

Proof of claim: Suppose the required minimum number of $K < 990$. Let a_i be the number of keys that the i^{th} member has; and let k_i be the number of members having the key for the i^{th} room. Then we must have

$$\sum_{i=1}^{100} a_i = K = \sum_{i=1}^{90} k_i.$$

Now at least one k_i has to be less than 11. Otherwise $\sum_{i=1}^{90} k_i = K \geq 990$.

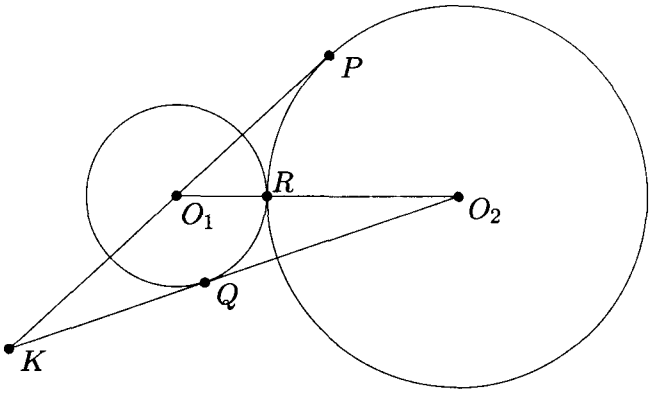
We may assume that $k_1 \leq 10$ and hence the first room key is at most with ten members and we may assume that they are the first ten members. In the remaining 90 members none of them has a key for the first room and this violates condition (a). Hence any $k < 990$ will not work and this proves the claim.

To repeat what others have said, requires education, to challenge it, requires brains.

Mary Pettibone Poole

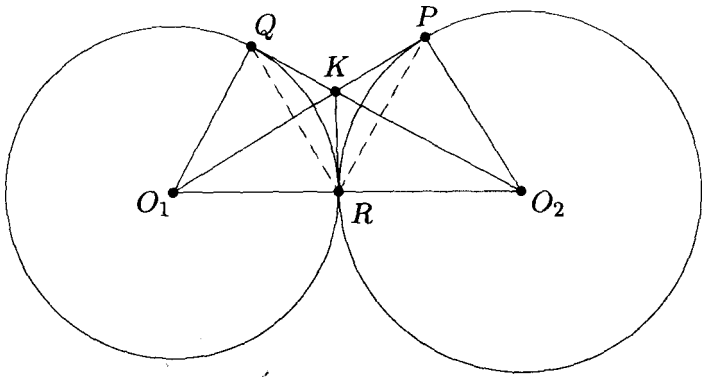
1. Let Γ_1 and Γ_2 be two circles touching each other externally at R. Let l_1 be a line which is tangent to Γ_2 at P and passing through the center O_1 of Γ_1. Similarly, let l_2 be a line which is tangent to Γ_2 at Q and passing through the center O_2 of Γ_2. Suppose l_1 and l_2 are not parallel and intersect at K. If $KP = KQ$, prove that the triangle PQR is equilateral.

Solution: Suppose that P and Q lie on the opposite sides of line joining O_1 and O_2. By symmetry we may assume that the configuration is as shown in the figure below. Then we have $KP > KO_1 > KQ$ since KO_1 is the hypotenuse of triangle KQO_1. This is a contradiction to the given assumption, and therefore P and Q lie on the same side of the line joining O_1 and O_2.



Since $KP = KQ$ it follows that K lies on the radical axis of the given circles, which is the common tangent at R. Therefore $KP = KQ = KR$ and hence K is the cirumcenter of $\triangle PQR$.

On the other hand, $\triangle KQO_1$ and $\triangle KRO_1$ are both right-angled triangles with $KQ = KR$ and $QO_1 = RO_1$, and hence the two triangles are congruent. Therefore $\widehat{QKO_1} = \widehat{RKO_1}$,



so KO_1, and hence PK is perpendicular to QR. Similarly, QK is perpendicular to PR, so it follows that K is the orthocenter of $\triangle PQR$. Hence we have that $\triangle PQR$ is equilateral.

Alternate Solution: We again rule out the possibility that P and Q are on the opposite side of the line joining O_1O_2, and assume that they are on the same side.

Observe that $\triangle KPO_2$ is congruent to $\triangle KQO_1$ (since $KP = KQ$). Therefore $O_1P = O_2Q = r$ (say). In $\triangle O_1O_2Q$, we have $O_1QO_2 = \pi/2$ and R is the midpoint of the hypotenuse, so $RQ = RO_1 = r$. Therefore $\triangle O_1RQ$ is equilateral, so $\widehat{QRO_1} = \pi/3$. Similarly, $PR = r$ and $\widehat{PRO_2} = \pi/3$, hence $\widehat{PRQ} = \pi/3$. Since $PR = QR$ it follows that $\triangle PQR$ is equilateral. □

2. Find all positive integers m, n, and primes $p \geq 5$ such that

$$m(4m^2 + m + 12) = 3(p^n - 1).$$

Solution: Rewriting the given equation we have

$$4m^3 + m^2 + 12m + 3 = 3p^n.$$

The left hand side equals $(4m + 1)(m^2 + 3)$.

Suppose that $(4m+1, m^2+3) = 1$. Then $(4m+1, m^2+3) = (3p^n, 1), (3, p^n), (p^n, 3)$ or $(1, 3p^n)$, a contradiction since $4m+1, m^2+3 \geq 4$. Therefore $(4m+1, m^2+3) > 1$.

Since $4m+1$ is odd we have $(4m+1, m^2+3) = (4m+1, 16m^2+48) = (4m+1, 49) = 7$ or 49.

This proves that $p = 7$, and $4m+1 = 3 \cdot 7^l$ or 7^k for some natural number k. If $(4m+1, 49) = 7$ then we have $k = 1$ and $4m+1 = 21$ which does not lead to a solution. Therefore $(4m+1, m^2+3) = 49$.

If 7^3 divides $4m+1$ then it does not divide m^2+3, so we get $m^2+3 \leq 3 \cdot 7^2 < 7^3 \leq 4m+1$. This implies $(m-2)^2 < 2$, so $m \leq 3$, which does not lead to a solution. Therefore we have $4m+1 = 49$ which implies $m = 12$ and $n = 4$. Thus $(m, n, p) = (12, 4, 7)$ is the only solution. □

3. Let a, b, c, d be positive integers such that $a \geq b \geq c \geq d$. Prove that the equation $x^4 - ax^3 - bx^2 - cx - d = 0$ has no integer solution.

Solution: Suppose that m is an integer root of $x^4 - ax^3 - bx^2 - cx - d = 0$. As $d \neq 0$, we have $m \neq 0$. Suppose now that $m > 0$. Then $m^4 - am^3 = bm^3 + cm + d > 0$ and hence $m > a \geq d$. On the other hand $d = m(m^3 - am^2 - bm - c)$ and hence m divides d, so $m \leq d$, a contradiction. If $m < 0$, then writing $n = -m > 0$ we have $n^4 + an^3 - bn^2 + cn - d = n^4 + n^2(an - b) + (cn - d) > 0$, a contradiction. This proves that the given polynomial has no integer roots. □

4. Let n be a positive integer. Call a nonempty subset S of $\{1, 2, \cdots, n\}$ good if the arithmetic mean of the elements of S is also an integer. Further let t_n denote the number of good subsets of $\{1, 2, \cdots, n\}$. Prove that t_n and n are both odd or both even.

Solution: We show that $T_n - n$ is even. Note that the subsets $\{1\}, \{2\}, \cdots, \{n\}$ are good. Among the other good subsets, let A be the collection of subsets with an integer average which belongs to the subset, and let B be the collection of subsets with an integer average which is not a member of the subset. Then there is a bijection between A and B, because removing the average takes a member of A to a member of B; and including the average in a member of B takes it to its inverse. So $T_n - n = |A| + |B|$ is even. □

Alternate Solution: Let $S = \{1, 2, \cdots, n\}$. For a subset A of S, let $\overline{A} = \{n + 1 - a | a \in A\}$. We call a subset A symmetric if $\overline{A} = A$. Note that the arithmetic mean of a symmetric subset is $(n + 1)/2$. Therefore, if n is even, then there are no symmetric good subsets, while if n is odd then every symmetric subset is good.

If A is a proper good subset of S, then so is $\overline{A}$. Therefore, all the good subsets that are not symmetric can be paired. If n is even then this proves that t_n is even. If n is odd, we have to show that there are odd number of symmetric subsets. For this, we note that a symmetric subset contains the element $(n+1)/2$ if and only if it has odd number of elements. Therefore, for any natural number k, the number of symmetric subsets of size $2k$ equals the number of symmetric subsets of size $2k + 1$. The result now follows since there is exactly one symmetric subset with only one element. □

5. In an acute triangle ABC, O is the circumcenter, H is the orthocenter and G is the centroid. Let OD be perpendicular to BC and HE be perpendicular to CA, with D on BC and E on CA. Let F be the midpoint of AB. Suppose the areas of triangles ODC, HEA and GFB are equal. Find all the possible values of $\widehat{C}$.

Solution: Let R be the circumradius of $\triangle ABC$ and $\triangle$ its area. We have $OD = R\cos A$ and $DC = \frac{a}{2}$, so

$$[ODC] = \frac{1}{2} \cdot OD \cdot DC = \frac{1}{2} \cdot R\cos A \cdot R\sin A = \frac{1}{2}R^2 \sin A \cos A. \tag{1}$$

Again $HE = 2R\cos C \cos A$ and $EA = c\cos A$. Hence

$$[HEA] = \frac{1}{2} \cdot HE \cdot EA = \frac{1}{2} \cdot 2R\cos C \cos A \cdot C\cos A$$
$$= 2R^2 \sin C \cos C \cos^2 A. \tag{2}$$

Further

$$[GFB] = \frac{\triangle}{6} = \frac{1}{6} \cdot 2R^2 \sin A \sin B \sin C = \frac{1}{3}R^2 \sin A \sin B \sin C. \tag{3}$$

Equating (1) and (2) we get $\tan A = 4\sin C \cos C$. And equating (1) and (3), and using this relation we get

$$\begin{aligned} 3\cos A &= 2\sin B \sin C = 2\sin(C + A)\sin C \\ &= 2(\sin C + \cos C \tan A)\sin C \cos A \\ &= 2\sin^2 C(1 + 4\cos^2 C)\cos A. \end{aligned}$$

Since $\cos A \neq 0$ we get $3 = 2t(-4t + 5)$ where $t = \sin^2 C$. This implies $(4t-3)(2t-1) = 0$ and therefore, since $\sin C > 0$, we get $\sin C = \sqrt{3}/2$ or $\sin C = 1/\sqrt{2}$. Because $\triangle ABC$ is acute, it follows that $\widehat{C} = \pi/3$ or $\pi/4$.

We observe that the given conditions are satisfied in an equilateral triangle, so $\widehat{C} = \pi/3$ is a possibility. Also, the conditions are satisfied in a triangle where $\widehat{C} = \pi/4$, $\widehat{A} = \tan^{-1} 2$ and $\widehat{B} = \tan^{-1} 3$. Therefore $\widehat{C} = \pi/4$ is also a possibility.

Thus the two possible values of $\widehat{C}$ are $\pi/3$ and $\pi/4$. □

6. Let a, b, c, x, y, z be positive real numbers such that $a+b+c = x + y + z$ and $abc = xyz$. Further, suppose that $a \leq x < y < z \leq c$ and $a < b < c$. Prove that $a = x$, $b = y$ and $c = z$.

Solution: Let

$$f(t) = (t - x)(t - y)(t - z) - (t - a)(t - b)(t - c).$$

Then $f(t) = kt$ for some constant k. Note that $ka = f(a) = (a - x)(a - y)(a - z) \leq 0$ and hence $k \leq 0$. Similarly, $kc = f(c) = (c - x)(c - y)(c - z) \geq 0$ and hence $k \geq 0$. Combining the two, it follows that $k = 0$ and that $f(a) = f(c) = 0$. These equalities imply that $a = x$ and $c = z$, and then it also follows that $b = y$. □

There is no branch of mathematics, however abstract which may not one day be applied to phenomena of the real world.

Nikolai Lobachevsky

Association Activities:

1. The annual account for 2012-13 has been audited and scrutinized by the executive committee which has approved the budget for 2013-14 with that as base.
2. NMTC 45 screening test took place on 31.08.2013. There were a few slips in the question papers due to computer incompatibility brought to our notice very late. Hence it was decided to delete those questions from evaluation so that the children may not be affected, particularly with negative marking for wrong answers. The number of participating institutions has increased to 621 with 93502 participants this year.
3. Week end classes announced saw very few seekers and hence it has been abandoned.
4. We have proposed web-based classes for concept clarification up to class 10 for which a note is displayed in our website `amtionline.com`
5. Conference call notice has already been dispatched to members by August end. It will be at KOCHI on 10, 11, 12 January 2014.
6. The annual workshops for teachers and students as last year will take place in May 2014 for which details will be displayed in website in January 2014.
7. Proposals are sent to NCSTC-DST for three more workshops for teachers this year (2013-14). Response in awaited.
8. Corporation of Chennai has approached us to introduce Olympiad to their children. After a test for about 400 children 40 have been selected and they are to attend 20 orientation classes. Five of the classes are already over. Based on that we hope that at least about 10 may rise to the demand of Olympiad.

Data regarding the 45th National Mathematics Talents Contest

NUMBER OF INSTITUTIONS REGISTERED: 622

States and the no. of institutions registered:		Languages represented & catered:	LEVELS					
State	No.		Primary	Sub-junior	junior	Inter	Senior	Total
Andhra Pradesh	*408*	***English***	*18386*	*30836*	*34280*	*9026*	*323*	*92851*
Delhi	*4*	***Gujarati***	*40*	*50*	*14*	*33*	*0*	*137*
Goa	*2*	***Hindi***	*1*	*0*	*1*	*103*	*0*	*105*
Gujarat	*11*	***Kannada***	*0*	*0*	*6*	*0*	*0*	*6*
Haryana	*6*	***Malayalam***	*26*	*32*	*64*	*1*	*0*	*123*
Karnataka	*32*	***Maratti***	*29*	*49*	*55*	*14*	*0*	*147*
Kerala	*9*	***Tamil***	*0*	*0*	*1*	*0*	*0*	*1*
Maharashtra	*42*	***Telugu***	*0*	*0*	*0*	*0*	*0*	*0*
M.P	*1*	***Total***						*93370*
Orisha	*2*							
Pondicherry	*2*							
Rajasthan	*10*	**OPEN QUOTA** (Based on newspaper notification						
Tamil Nadu	*87*							
U.P	*6*							
		English	*40*	*68*	*120*	*88*	*3*	*320*
		others						
		Total						*320*
Total	*622*	***Open quota (States and the no. of institutions registered)***						
		Andhra Pradesh	*19*	*Kerala*		*21*		
		Bihar	*1*	*Maharashtra*		*27*		
		Delhi	*4*	*Rajashtan*		*1*		
		Gujarat	*15*	*Tamilnadu*		*115*		
		Haryana	*2*	*U.P*		*55*		
		H.P	*1*	*West Bengal*		*5*		
		Karnataka	*54*					
				Total		*320*		
		Second Level Candidates						
		Finally selected						

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